INTERVIEW WITH SAHARON SHELAH

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In August 2000 we participated in the conference **Mathematical Challenges of the 21$^{\mathbf{th}}$ Century** *on the 100th anniversary of the famous conference in Paris, in which Hilbert exposed 23 open problems in Mathematics. Do you believe the history of Mathematics to be a part of Mathematics?*

I'd rather say that the history of Mathematics is part and parcel of History. The history of Mathematics is a very interesting and worthwhile intellectual area, but it does not contribute directly to mathematical research. In several cases, historians wrote on Mathematics and were criticized for not understanding it, even though it was ancient Mathematics. This is not surprising since according to the traditional education of historians, you cannot expect them to know mathematics properly. On the other hand, historians would say that we mathematicians understand Mathematics but not History. There have been some interesting phenomena, such as the simultaneous discovery of differential and integral calculus by Newton and Leibniz, so one could assume that the time was ripe for it. But perhaps we should not exaggerate and suppose that history is predetermined and the only question is which individual will carry out the task. There are counter examples, such as the general method discovered by Archimedes to calculate areas and volumes by considering their center of gravity. Archimedes did not succeed in proving his method, perhaps because of lack of appropriate symbols and tools, but he used it in solving various questions to which he subsequently gave *rigorous* proofs. He described the method in a separate book, however. It fell into oblivion, and both Newton and Leibniz arrived to a similar point. Around 1900, a copy of Archimedes' book was found.

About 500 million years ago, only monocellular organisms lived upon the face of Earth and then, along with the so-called "Cambrium explosion", polycellular creatures commenced to emerge. But when we say 'along with' we mean plus-minus some millions of years, so several

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thousand years of the History of Mathematics is not such a long time in terms of the Earth's age.

You can regard different questions with different scales. Maybe Mathematics will develop differently in the flow of time. One can propose such plausible possibilities, but as humans are not perfect, and in order to have a good understanding, we should give predictions and verify them. If we have no such feed-backs we may tend to give nonsense opinions. For example, to the question: *Could intelligent beings develop another Mathematics or Physics?*, we probably will get convincing answers only if we meet different cultures. I feel that Mathematics, of all things, would be much easier to understand in a cross–cultural perspective than other cultural elements. One could try, as an exercise, to develop another kind of Mathematics, but the question is where to start? Should one give up common logic?

There have been different cultures on Earth and each one developed Mathematics to some extent. We have documents from roughly 5000 years. As far as we know, the development of Mathematics with precise demonstrations appeared first in the Greek culture.

What made you devote yourself to Mathematics?

While I was still in primary school, I knew I wanted to become a scientist, but as Mathematics was taught, it did not attract me especially. I was very interested, on the other hand, in Physics and Biology and I read popular scientific books. But when I reached the ninth grade I began studying geometry and my eyes opened to that beauty - a system of demonstrations and theorems based on a very small number of axioms which impressed me and captivated me. On the other hand laboratories were not to my taste and vice versa. So by the age of 15 I knew my desire to be a mathematician. Later I read Abraham Halevy Fraenkel's (1891-1965) book 'An Introduction to Mathematics'. I have seen many popular books on Mathematics in English, but I think Fraenkel's book is better, and indeed he help coment my choice of Mathematics.

As a Mathematician, what do you do in your office all day?

Since I have succeeded in demonstrating a substantial number of theorems, I have also a lot of work completing and correcting the demos. As I write, I have a secretary typing (I did have a lot of troubles concerning this) and I have to proof-read a lot. I write and make corrections, send to the typist, get it back and revise it again and again.

A great amount of time is used to verify what I wrote. If it is not accurate or utterly wrong, I ask myself what went wrong. I tell myself: there must be a hole somewhere, so I try to fill it. Or perhaps there is a wrong way of looking at things or a mistake of understanding. Therefore one must correct or change or even throw everything and start all over again, or leave the whole matter. Many times what I wrote first was right, but the following steps were not, therefore one should check everything cautiously. Sometimes, what seems to be a tiny inaccuracy leads to the conclusion that the method is inadequate. I have a primeval picture of my goal. Let us assume that I have heard of a problem and it seems alike to problems that I know how to resolve, provided we change some elements. It often happens that, having thought of a problem without solving it, I get a new idea. But if you only think or even talk but not sit down and write, you do not see all the defaults in your original idea. Writing does not provide a 100% assurance, but it forces you to be precise. I write something and then I get stuck and I ask myself perhaps it might work in another direction? As if you were pulling the blanket to one part and then another part is exposed. You should see that all parts are integrated into some kind of completeness.

Indeed, sometimes you are happy in a moment of discovery. But then you find out, while checking up, that you were wrong. There has been a joy of discovery, but that is not enough, for you should write and check all the details. My office is full of drafts, which turned to be nonsense.

When you make a mathematical discovery, do you think it exists also outside the human mind?

It is difficult for Man to describe the world without him. We can hardly imagine our death. Otherwise probably we would not be able to concentrate. Therefore we can also hardly imagine that there exists something outside of us. But under axioms and logical rules, it seems that Mathematics is absolute. Plato's kind of argument according to which Mathematics is an idea which we discover will always be controversial, but to questions such as: is a given solution to a mathematical problem the right one, the answer is clear–cut. True, we can have only evidence and not an absolute proof of the existence of Mathematics outside of man, because we only act in a human reality. Yet, if one day we meet the little green creatures and speak with them, we might understand a lot more. Most mathematicians, in order to work, must at least pretend they are Platonists. If I felt that what I do is only playing games with symbols without any other meaning, I would probably not do Mathematics.

In the thirties, Kurt Gödel (1906-1974) demonstrated that in every complicated enough mathematical field there is a problem which cannot *be solved by following the axioms of that field. What did that discovery do to 20th century Mathematics?*

I suppose that most mathematicians who were doing research in some particular field and were aware of Gödel's theorem thought it is interesting but with no direct concern with the problems which really bothered them. Some people were utterly shocked and lost interest in some part of Mathematics, but truth is important even when it is unpleasant. During the conference *Mathematical Challenges of the 21*th *Century* I presented the question whether we can find a parallel method to demonstrate independence in Number Theory, as Paul Cohen did in Set Theory. Hilbert's first question in Paris was one that Cantor had asked too and we will return to it later. It has been answered, but usually if you give a correct answer to a good question, seven other questions emerge instead.

You mean for example the demonstration that Goldbach's or Riemann's hypotheses are independent from the number theory axioms?

If that can be proved in Mathematics, it will be an astounding achievement. One could add more subtle distinctions of the hypotheses, but we will not give them in detail here.

> **Goldbach's hypothesis:** *Any even number can be described as a sum of two primes. For example, 17+13=30, 13+11=24.*

As yet, no proof has been found, nor has been found a counter-example.

> **Riemann's hypothesis:** *The roots of* ζ*–function are in the line of complex numbers whose real part is 1/2.*

$$\zeta(x) = \sum_{n=1}^{\infty} (1/n^x) \qquad \{\text{real}(x) > 1\}.$$

In 1976 it has been proved by the computer that any map can be coloured with 4 colours so that no country bears the same colour. Thus, a 100 years conjecture has been demonstrated. Do you think that in another 100 years too, mathematicians will search for axioms, definitions and theorems? Since computers may do it better?

Many mathematicians were quite perplex and did not know how to treat the 4-colour problem. When you think about a mathematical problem, you do not just sit there expecting a voice to tell you if the theorem is true or not. You want to understand the problem in depth. Take for example Fermat's conjecture, whose understanding led to a profound understanding in Algebraic Number Theory.

For the 4 color conjecture, the computer helped us to prove it by dividing to great number of cases. We feel, perhaps, like Alice in "Through the Looking Glass". When the red queen asks Alice if she can sum up, Alice answers that she does. Then, the queen asks her swiftly how much is $1+1+1+1+1+1+1+1+\ldots$, and she is not able to answer even though she can sum up.

There may exist in some future an artificial intelligence which will keep Man in natural reserves, as we do with chimps. We may wonder if there is any point to have people making intellectual activity in that time. There may also exist, however, a situation in which people's brain and computers are not separated from each other. And perhaps then we will be discovered by little green creatures who will legislate to preserve mankind?

We may ask, can we obtain a plausible answer to each question, which we find interesting?

Once upon a time people used to beat with their hands, then they learnt to throw stones, then to shoot guns. But there will always be a limit to the range they can attain. Likewise, by using computers we can check more cases, but there will always be a limit. Gödel's incompleteness theorem ensures us that there will always be new questions that we cannot solve in a given level of axioms and computing power. It may be, however, that our Mathematics is but the opening stage to another Mathematics, and then another kind of problems, of a higher level, will emerge, which is akin to our situation as human beings.

Is it possible that the special combination of Logic and Paradoxes, as it is expressed in Lewis Carroll's book, become in the future the basis to a new form of Mathematics?

Archimedes created a new Mathematics but he had no direct heirs. Maybe the lack of adequate symbols contributed to the delay in evolution. Of course, the situation of culture and society had an influence too. If you look at mathematical formulae and write them in conventional language, the result will be much less clear. I admire Carroll's writing, but it includes no mathematical breakthrough. He did not try to prove anything new but presented in an amusing way some paradoxes, which is not, in itself, a new mathematical content. On the other hand, if books in Mathematics were written like Lewis Carroll used to write, surely more people would do Mathematics.

The Indian mathematician Srinivasa Ramanujan (1887-1920) was undoubtedly an extraordinary phenomenon in Mathematics. We know for example that he discovered 4000 formulae without being able to prove them. How did he manage to do it, in your opinion?

I have read Ramanujan's biography, which was not written by a mathematician, and I do not think there is any logical problem about his achievement. Many people have non-formalized ways of demonstration, and so it is not clear when they are valid. Certainly, there are cases in which the results obtained are wrong, yet in the framework of the questions they asked it did work out well. Ramanujan had probably some sort of fundamental understanding, which allowed him to work. One should also recall that he studied in India, which was similar to studying in Great Britain some 40 years earlier, namely Newton's methods without the precision of epsilon and delta. What mattered was to get a right result, less important was the method. There were also some theorems about the distribution of primes which Hardy showed him to be wrong. Nevertheless, he was one of the most talented individuals in the history of Mathematics. It would be quite interesting to try and reconstruct the way he thought.

Anyway, a physicist also checks his discoveries in the light of reality. For example, if the solution of an equation leads to the conclusion that during the Big Bang, there existed not enough matter, then he will know that there is some mistake, because the solution should correspond to reality to a rather high degree, so Physics has control criteria. It was true in number theory in Fermat's day, and to some extent in Ramanujan's research, but when you get to theorems for which there exists no direct feed-back then clearly without precision you will go nowhere.

Is there a part of your field of research which you can describe?

I am interested in the first problem that Hilbert exposed in 1900, though it has been solved by Gödel in the late forties (one half) and the other half by Paul Cohen in the sixties. If we managed to get Cantor out of his tomb, he would most certainly not be happy with the solutions to his questions, because they are meta-mathematic.

But we would better begin by explaining the discoveries made by Cantor and others. His view was, as a matter of fact, very ancient and simple. We know that primitive peoples do not have large numbers, certainly not beyond 40. So how do they trade? Very simple: they trade one for one. Two sets of sheep are equivalent if there is a one to one correspondence between them. I shall trade twenty sheep for ten gems in ten steps: Each time I trade two sheep for a gem. Likewise, if someone is the depositary of a herd of sheep, in order to check that all of them were given back, they would put for each sheep a stone in a jar, and then seal it. When the sheep got back, they would open the jar and check the correspondence between sheep and stone.

And back to Cantor. He said: let us forget about the finiteness of the set and use the same test of a one-to-one correspondence to see if two sets are equivalent. In this case, for example, the set of positive natural numbers is equivalent – or in terms of set theory it has the same cardinality, or the same number of elements – to the set of even integers, since you can trade any natural number for a number two times greater.

$$\begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & \dots \\ | & | & | & | & | & | & | & | & | & | & \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 16 & 18 & 20 & \dots \end{array}$$

Similarly, we define an order. The number of elements in set a A is smaller than or equal to the number of elements in a set B if there is a correspondence from A into B, that is there is one-to-one correspondence between the elements of A and the elements of some subset of B. But if the number of elements of A is smaller or equal to the number of elements of B and vice versa, then those sets are equivalent, however this require a proof. It is also possible to show that the set of positive rationals, namely the numbers which are quotients of two positive integers, is equivalent to the set of natural numbers, since you can arrange the (positive) rationals in one sequence: first you count all the numbers in which the sum of the denominator and the numerator is 1, then those where the sum is 2, etc. If you obtain a number which is listed already – you omit it.

It was very surprising to discover that the number of points on the line is equivalent to the infinite number of points on the plane, the idea is that you can split the decimal representation of a number into two numbers according to the figures in the even and in the odd positions:

$$0.x_1x_2x_3x_4x_5x_6\dots \quad \longleftrightarrow \quad (0.x_1x_3x_5\dots):(0.x_2x_4x_6\dots).$$

It would have been sensible then to think that all infinite sets are equivalent. But it does not turn out to be the case since Cantor has shown, for example, that you cannot establish a one-to-one correspondence between the natural numbers and the real numbers (index by the natural numbers) . For any infinite sequence of real numbers, there are real numbers that do not appear in the sequence, and therefore there are strictly more real numbers than natural numbers. For example, any number b such as its n^{th} digit is different from the n^{th} digit of a_n, will not be in the sequence.

$$\begin{array}{lllllll} a_1 & = & 0. & \mathbf{x_1^1} & x_2^1 & x_3^1 & x_4^1 & \dots \\ a_2 & = & 0. & x_1^2 & \mathbf{x_2^2} & x_3^2 & x_4^2 & \dots \\ a_3 & = & 0. & x_1^3 & x_2^3 & \mathbf{x_3^3} & x_4^3 & \dots \\ a_4 & = & 0. & x_1^4 & x_2^4 & x_3^4 & \mathbf{x_4^4} & \dots \\ & & & & \dots & & & \end{array}$$

Since it turned out that there are several levels of infinite, it became necessary to give them names, and Cantor began to use the $\aleph$ symbol. He denotes by $\aleph_0$ the cardinality of the set of natural numbers. It turned out that every infinity has an immediately greater one, a successor. So $\aleph_1$ is the successor of $\aleph_0$, $\aleph_2$ is the successor of $\aleph_1$ etc. The limit of $\aleph_n$'s is called $\aleph_\omega$.

$$\aleph_0, \aleph_1, \aleph_2, \aleph_3, \aleph_4, \dots, \aleph_n, \dots, \aleph_\omega.$$

The first number which has $\aleph_4$ numbers below it is symbolized by $\aleph_{\omega_4}$. Later Zermelo proved that any two sets can be compared.

What happens to the arithmetic operations with infinite numbers ?

You can define naturally a sum of infinite cardinals of two disjoint sets. The number of elements of the union is the sum of elements in both sets. There is also a natural definition to multiplication: From the sets A, B we can define the set C of pairs (a, b), where a is from the set A and b is from the set B. Now, the number of elements of the set C is the number of elements of A multiplied by the number of elements of B. (well, we have to prove that the operation is well defined i.e does not depend on A and B just on the number of their elements)

What happens to division or subtraction ?

There is no definition to that because you can not cancel. For example: any infinite number plus one is equal to itself, that is: $x+1 = x$. If the rule of cancellation was correct, we would obtain $0 = 1$. For finite sets we get the normal operations. It is not enough to define, we have to prove that the operations are well defined.

Does the arithmetic of infinite numbers have rules like the finitary ones?

They look wonderful! Any schoolboy would prefer them to the normal rules. From the beginning of the 20th century we know that the sum of two infinite numbers, or two numbers of which at least one is infinite, is the greater of the two, similarly for the product when both are not zero.

What about the exponent?

One can define a multiplication of possibly infinitely many numbers and especially exponentiation. For instance, 2^x is the number of subsets of a set A with x elements. For example: the set $\{1, 2, 3\}$ has 8 subsets as follows: $\{\}, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}$.

Are the operations concerning exponents as simple, too?

No, because Cantor showed that 2^x is always greater than x. We have then two operations which increase x: the successor number of x and 2^x. Our life would be simple if there were just one such operation, namely if both operations were equal. You can concentrate on the most simple case for the natural numbers, which is Cantor's Continuum Hypothesis and the first problem exposed by Hilbert in Paris. The question was: is any infinite set of real numbers either equivalent to the set of natural numbers or to the set of reals?

A way to express this question is: Does

$$\prod_n 2^n = \aleph_1 \ ?$$

At the end of the thirties Gödel showed that you cannot contradict the Continuum Hypothesis. In the sixties, Paul Cohen showed that you cannot prove the Continuum Hypothesis. It means that the Continuum Hypothesis does not depend on the axioms of set theory which are called ZFC, after Zermelo and Fraenkel. So you can assume it is right and you can assume it is wrong.

It reminds us of the axiom of parallels, which has been proved not to depend on the axioms of geometry. Don't you have a strange feeling when you deal with such questions, do you touch infinity?

I am very interested in it, and it is nice that one can say something on infinity after all. A lot has been done by Solovay, Easton, Galvin, Hajnal, Magidor, Woodin, Gitik and mitchal in the world in different directions. I'll tell one more thing. I tried to show, in a book published in the last decade, that you can prove many things on infinity on the basis of the ZFC set theory if you just ask the right questions.

To simplify, let us look at the simplest case left, the product

$$\aleph_0 \times \aleph_1 \times \aleph_2 \times \ldots \times \aleph_n \times \ldots = (\aleph_\omega)^{\aleph_0}.$$

This number is equal or greater to the number of reals, $2^{\aleph_0}$. Let us suppose that the number of reals is equivalent to $\aleph_n$ for some natural n. How big can the product be? Of course, there are infinite number of infinities smaller than the product, but not too much, the number of smaller numbers is smaller than $\aleph_{\omega_4}$:

$$\prod_n \aleph_n \leq \aleph_{\omega_4} + 2^{\aleph_0}.$$

It sounds rather amazing, why 4 in a formula on infinities? Is it linked to the problem of colouring a map with 4 colours or perhaps to the 4 of the polynomial degree, which can be solved as Evariste Galois (1811-1832) discovered?

Many mathematicians have been surprised to see a link between 4 and infinite. Perhaps it is due only to my limitations, and in the future we will see that there is a lower limit, like $\aleph_{\omega_1}$. Anyway, it is one of the questions I have asked during my lecture "Logical dreams" in the conference *Mathematical Challenges of the 21*$^{\text{th}}$ *Century* in U.C.L.A.