# Complementary Mathematics
## *Various Degrees of the Numbers' Distinction*

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**We introduce new, elementary concepts that emerged from observing children learning mathematics. This representation is geometric in nature.**

## 1. Abstract

Euclidean geometry is based on the axiom that through every point outside a given line there is exactly one line parallel to that given line. But, as every student of mathematics knows, we obtain great benefit from imagining systems in which this axiom is not true. Exploiting our studies of how children learn mathematics, we suggest an augmentation of the concept of number.

Young children see numbers as simultaneously ordinal and cardinal. We discovered by working with children that there is yet another dimension to their conception of numbers that we call the ***degree of distinction.*** We relate this observation to the fact that children think of lines as atomic objects rather than as sets of points.

Through our investigations, we developed a new meta-conception of numbers, where the number becomes ***a bridge between the continuum and the discrete***. We discovered a new way to view numbers, which leads to a great variety of potentially clashing models.

These observations allow us to comprehend the difficulties young children encounter while studying the Cantorean concept of Infinity.

## 2. Introduction

Following are some thoughts expressed by five-year-old children while reasoning about points and lines:

**Nevo:** "We imagine some shape in our mind, and then we draw it. But we can draw shapes without lines and points and use them in order to create new shapes; all these shapes are in our imagination."

Nevo positions his source of creativity in his own mind; he discovers that an abstract idea is independent on any particular representation.

**Sivan:** "The shape is in our imagination; we can think about it, but it does not exist. So, we draw it, because we can see it within our hearts."

Sivan is aware of the difference between potential and existing things. In his experience, the innovation emerges from the heart.

**Tohar:** "If a point and a line are not friends and they do not reproduce, then they will remain few and nothing will be born."

Tohar is aware of the distinction between a point and a line. He realizes that new drawings can be created by associating them.

**Ofri**: "The point tries to catch the line, but it cannot catch it because the line is too high."

Ofri understands that a line has a height unreachable by a point.

These observations demonstrate that five-year-old children have natural creativity, an inner space. Developing this natural creativity is very important. By carefully listening to young children, deep notions may emerge. Using the independence of a point and a line observed by five-year-old children, we developed a new mathematical theory. While this is a preliminary work, we believe that further research will contribute to the development of the language of mathematics in general and to the theory of numbers in particular.

A point and a line are context-dependent concepts in Modern Mathematics. We define a point and a line as non-geometrical meta-concepts of membership. From this point of view, any mathematical framework is an organ of a meta-framework. Using this approach, we have provided an answer to Hilbert's 24$^{th}$ challenge. Quoting Hilbert's famous Paris 1900 lecture: *"The problems mentioned are merely samples of problems, yet they will suffice to show how rich, how manifold and how extensive the mathematical science of today is, and the question is urged upon us whether mathematics is doomed to the fate of those other sciences that have split up into separate branches, whose representatives scarcely understand one another and whose connection becomes ever more loose. I do not believe this nor wish it. Mathematical science is in my opinion an indivisible whole,* ***an organism*** *whose vitality is conditioned upon the connection of its parts."*

## 3. Locality and non-locality, basic terms and definitions

The set is a fundamental concept used in many branches of mathematics. Although not rigorously defined, a set can be thought of as a collection of members. If we expand the membership concept beyond the set/member relation, then new mathematical frameworks emerge. For example:

*or* is a logical OR connective.

*xor* is a logical EXCLUSIVE OR connective.

*and* is a logical AND connective.

= is "Equal to …".

≠ is "Not equal to …".

Urelement is not a set, but can be a member of a set.

Element is either a set or an urelement.

Sub-element is an element that defines another element.

∈ is a member of an element, but not necessarily a sub-element of an element.

∉ is not a member of an element.

***x*** and **A** are placeholders of an element.

**Definition 1:** If ***x*** **∈ A** *xor* ***x*** **∉ A,** then ***x*** is local.

**Definition 2:** If ***x*** **∈ A** *and* ***x*** **∉ A,** then ***x*** is non-local.

The point, ., is a local member. For example, if ***x*** = . and **A** = __ then _._ *xor* __ .

A set's member is a local member, that is, ***x*** **∈ A** *xor* ***x*** **∉ A**.

__ is also a non-local member. For example, if ***x*** = __ and **A** = . then _._ *and* _._

Since __ is a non-local urelement, then none of its members are its sub-elements. Strictly speaking, __ and any given local ***x*** are mutually independent, exactly like two axioms.

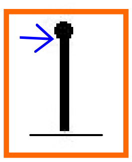
The membership concept has at least two different options:

**Option 1:** Membership between an element and its sub-elements (notated by $\epsilon$).

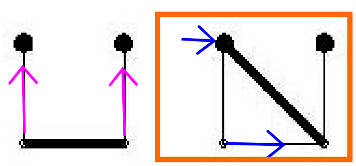
**Option 2:** Membership between an element and other elements, which are not necessarily its sub-elements (notated by $\boldsymbol{\in}$ ).

The first option is the standard set/member relation, while the second is called *bridging*. From a meta view on the mathematical language, each context-dependent axiomatic system is a local member that can be related to a non-local urelement. The bridging (represented by "|") between non locality (represented by "__") and locality (represented by one or more ".") is measured by symmetry, for example:

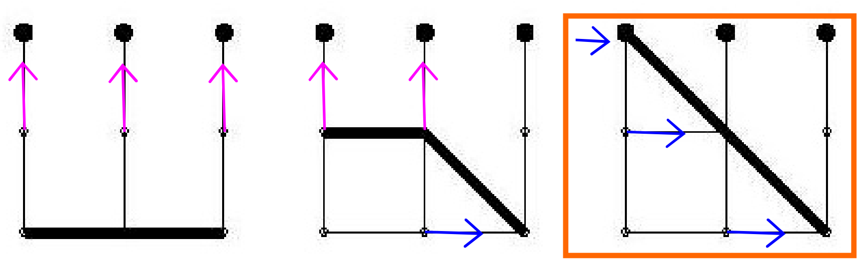
No bridging (nothing to be measured)



A single bridging (a broken symmetry, notated by →)



More than a single bridging that is measured by several symmetrical states, which exist between parallel symmetry (notated by ↑) and serial broken symmetry (notated by →).



Most modern mathematical frameworks are based on only broken symmetry, marked by orange rectangles, as a first-order property. We expand the research to both parallel and serial first-order symmetrical states in one organic meta-framework, based on bridging the local and non local.

Figure 1

Here is an example of a parallel/serial bridging between locality and non locality (bridging spaces 4 and 5):

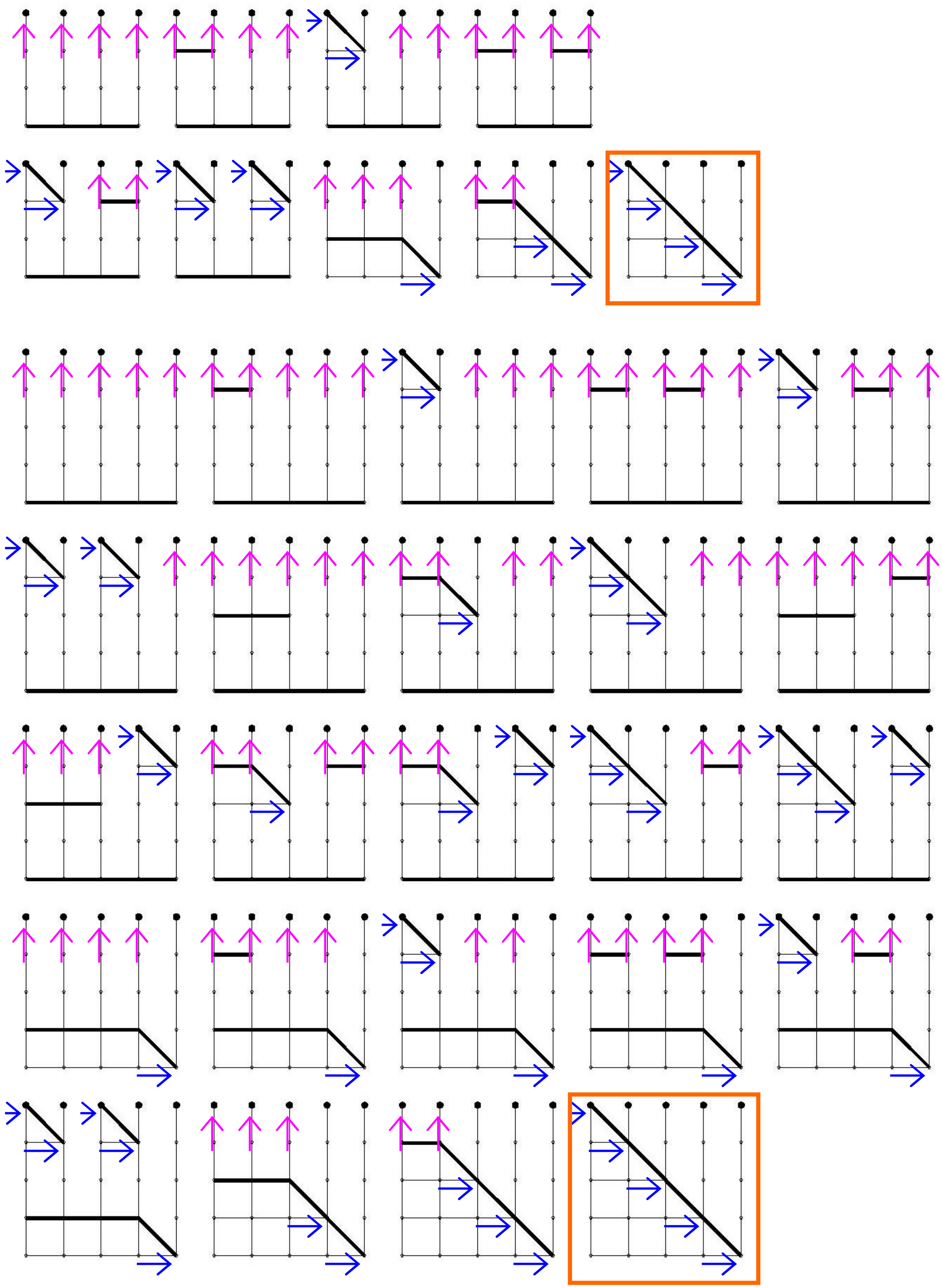


Figure 2

Axiomatic systems are based on mutually independent, self-evident true statements called *axioms*. If the context concept is equivalent to the set concept, then each distinct axiom is equivalent to a member of a broken-symmetry context. However, if the symmetry concept is used as the meta concept of the axiomatic approach, we provide a new mathematical framework that is not necessarily based on broken symmetry. In that case, uncertainty and redundancy are used as first-order concepts of a symmetry-based mathematical approach. Figures 1 and 2 define the non distinct (based on bridging between local and non local) as a first-order mathematical property.

**Definition 3:** Identity is a property of ***x*** *or* **A**, which allows distinguishing among them.

**Definition 4:** Copy is a duplication of a single identity.

**Definition 5:** If ***x*** *or* **A** have more than a single identity, then ***x*** *or* **A** are called *uncertain*.

**Definition 6:** If ***x*** *or* **A** have more than a single copy, then ***x*** *or* **A** are called *redundant*.

Here is an example of uncertainty and redundancy:

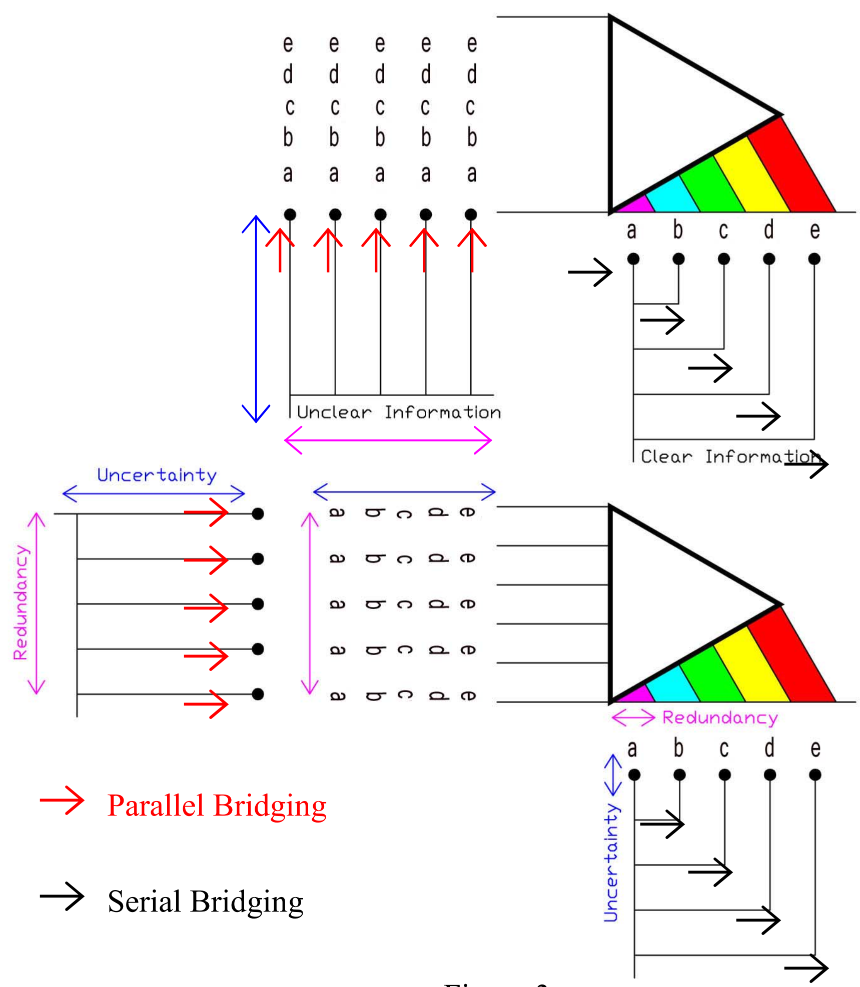



Figure 3

## 4. Organic Natural Numbers

First, here is the standard definition of the natural numbers.

**Definition 7:** *The set of all natural numbers* is the set $\mathbf{N} = \{x \mid x \in I \text{ for every inductive set } I\}$.

Thus, a set $x$ is a natural number iff it belongs to every inductive set. Each member of an inductive set is both a cardinal and an ordinal, because it is based on a broken-symmetry bridging, represented by each blue pattern in figure 4. Armed with symmetry as a first-order property, we define a bridging that cannot be both a cardinal and an ordinal, represented by each magenta pattern in figure 4. The products of the bridging between the local and the non-local are called *organic natural numbers*.

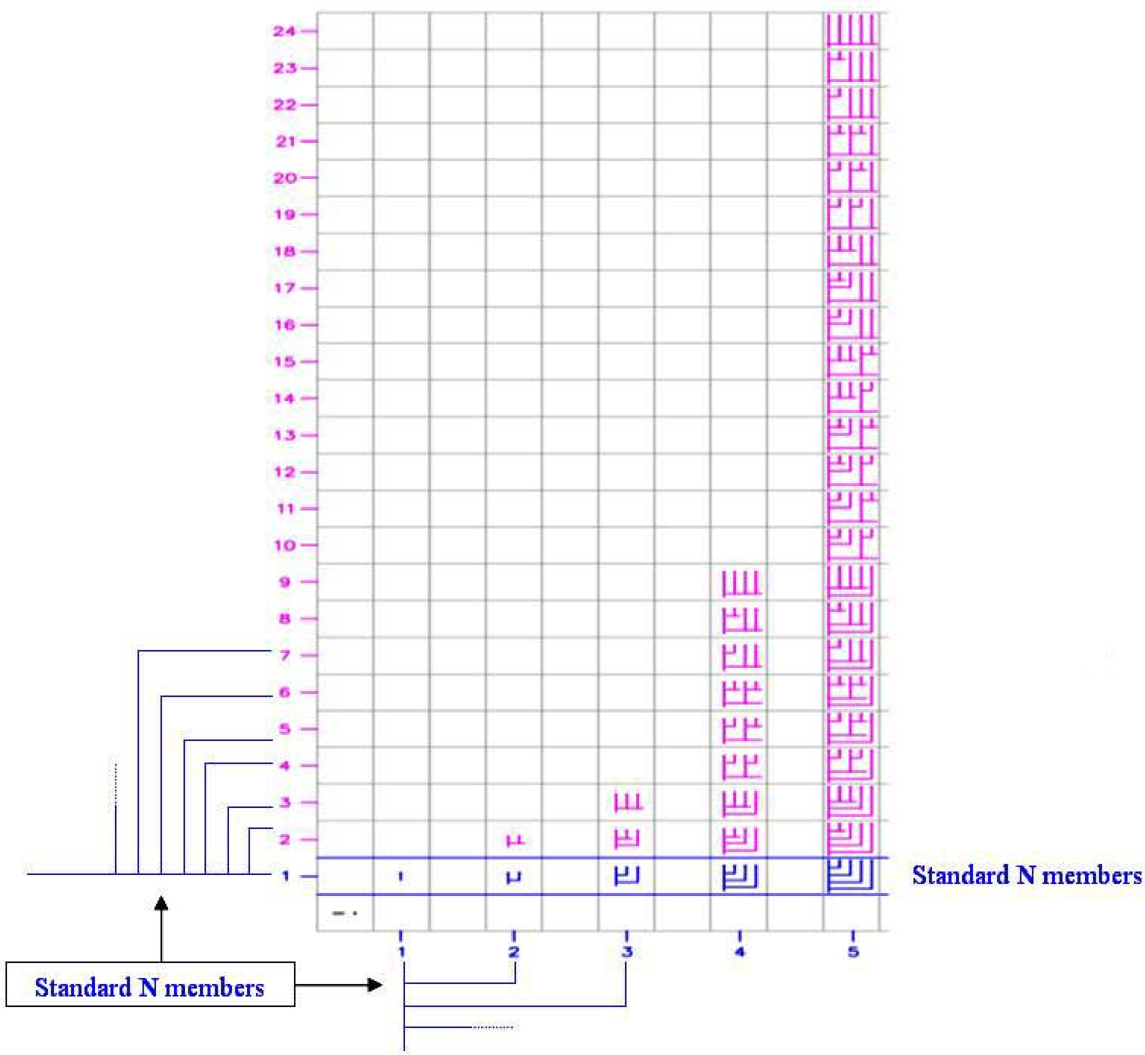



Figure 4

Each pattern in Figure 4 is both local/global state of the *organic natural numbers* system.

## 5. Locality, non-locality and the real-line

A *sequence* is a collection of elements ordered by some rule. A *continuum* is a property of a non-local urelement. If we define the real line as a non-local urelement, then no sequence is a continuum. By studying locality and non locality along the real line, we discovered a new type of numbers, the *non-local numbers*. For example,

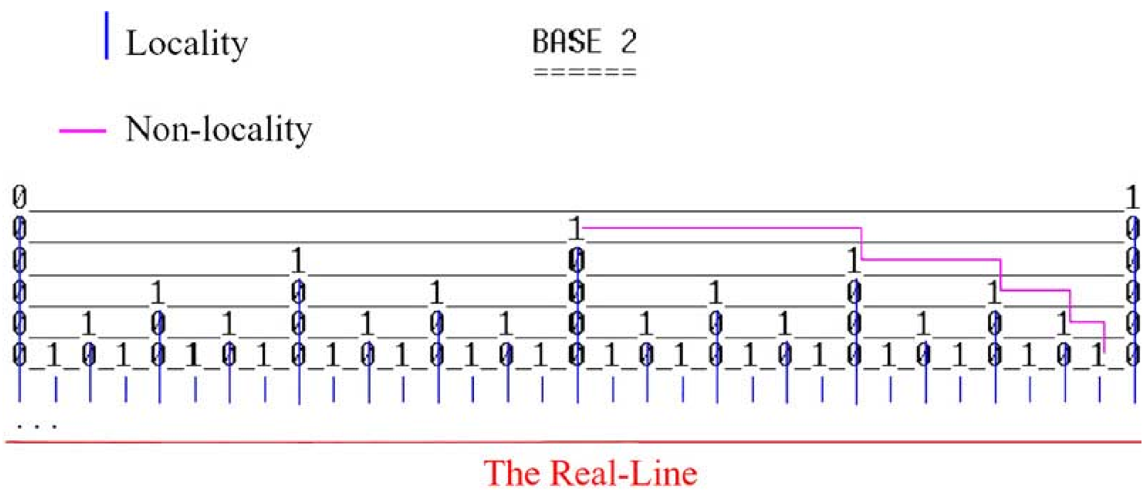



Figure 5

Figure 5 illustrates a proof that the number 0.111… is not a representation of the number ***1*** in base 2, but the non-local number ***0.111…*** < ***1***. Are there any numbers between ***0.111…*** and ***1***? Yes, for any given base ***n***>2, the non-local number ***0.kkk…*** where ***k***=***n***-1. For example:

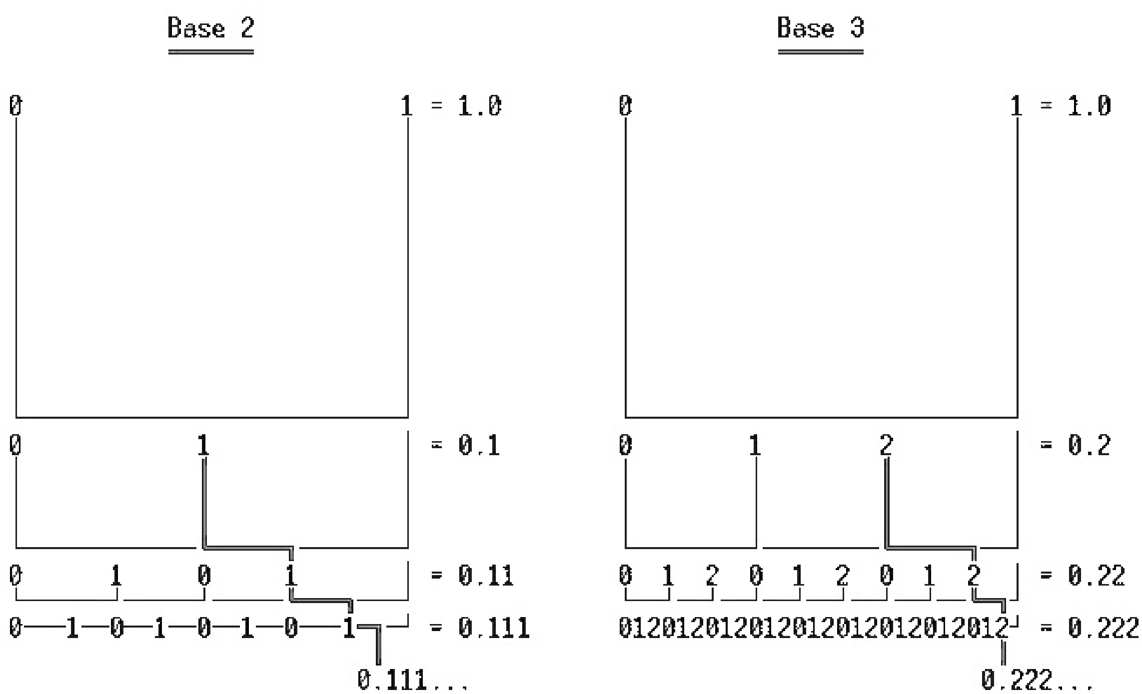



Figure 6

Figure 6 demonstrates that between any given pair of **R** members, which are local numbers, there exists a non-local number, whose exact location on the real line does not exist. A local number is not a limit of any non-local number, because local and non-local numbers are mutually independent. Furthermore, no set of infinitely many elements can reach the completeness of a non-local urelement. Thus, any non-finite set is an incomplete mathematical object, when compared to a non-local urelement.

A finite set is not impacted by this incompleteness, because it does not have the tendency to reach the real-line's completeness. In other words, a finite set has an accurate cardinal; however, a non-finite set does not have an accurate cardinal, because it is an incomplete mathematical element. From this new notion of the non finite, Cantor's second diagonal argument is understood as a proof of the incompleteness of a non-finite set. For example, assume a non-finite set composed of unique non-finite multisets, where each non-finite multiset has a unique order of empty and non-empty sets:

```
{
 {{ },{ },{ },{ },{ },...}
 {{#},{ },{ },{#},{ },...}
 {{ },{#},{#},{ },{ },...}
 {{#},{#},{ },{#},{#},...}
 {{ },{ },{#},{ },{ },...}
 ...
}
```

We can then define another unique, non-finite multiset, which is the non-finite, diagonal complementary multiset {**{#},{#},{},{ },{#},...**} that is added to our non-finite set of non finite multisets, etc., etc. … ad infinitum. From this point of view, the identity map of a non-finite set, composed of unique non-finite multisets, is incomplete; furthermore, its cardinality is unsatisfied.

We know that the Cantorean transfinite universe is an actual infinity, where the limiting "process" is a potential infinity. However, by the new notion of the non finite, we realize that since no non-finite set can reach the completeness of a non-local urelement, then only a non-local urelement is considered as an actual infinity. Here any non-finite set is no more than a potential infinity. These sets can have infinitely many potential infinities, but none reaches the completeness of a non-local urelement.

As an example, consider the 1-1 mapping between the non-finite sets ***A***={1,2,3,4,5,6,…} and ***B***={2,4,6,…}. These two sets have the same cardinality, even if ***B*** is a proper subset of ***A.*** Now check if there is any proportion between sets ***A*** and ***B***. By the following mapping, we can clearly see that ***B*** is a proper subset of ***A***:

```
1 2 3 4 5 6 …
↕ ↕ ↕ ↕ ↕ ↕
  2   4   6 …
```

But the following mapping demonstrates that sets ***A*** and ***B*** have the same cardinality:

```
1 2 3 4 5  6 …
↕ ↕ ↕ ↕ ↕  ↕
2 4 6 8 10 12 …
```

If some member is in set ***B***, it is also in set ***A***; since the number 12 is in set ***B*** in this particular example, then it is also in ***A***. Thus,

1 2 3 4 5 6 7 8 9 10 11 12 …
↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕
2 4 6 8 10 12 … etc. , … etc. ad infinitum.

From this new notion, we conclude that the result |***A***| = |***B***| is based on a partial picture of the 1-1 mapping,

1 2 3 4 5 6 …
↕ ↕ ↕ ↕ ↕ ↕
2 4 6 8 10 12 … which provides the illusion that |***A***| = |***B***|.

But the full picture of this 1-1 mapping is actually:

1 2 3 4 5 6 7 8 9 10 11 12 …
↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕ ↕
2 4 6 8 10 12 … etc. , … etc. ad infinitum.

Sets ***A*** and ***B*** are non-finite sets; applying the proportion concept, any non-finite set is no more than a potential-infinity, while distinguishing between locality and non-locality.

Let @ be a cardinal of a non-finite set, and let ***C*** and ***D*** be non-finite sets.

If |***C***| = @ and |***D***| = @-2^@, then |***C***| > |***D***| by 2^@. Applying the new notion of the non-finite, we have both non-finite sets and an arithmetic between non-finite sets, whose results are non-finite sets. For example,

By Cantor $\aleph_0 = \aleph_0+1$ , by the new notion @+1 > @.
By Cantor $\aleph_0 < 2$^$\aleph_0$ , by the new notion @ < 2^@.
By Cantor $\aleph_0$-2^$\aleph_0$ is undefined, by the new notion @-2^@ < @.
By Cantor 3^$\aleph_0$ = 2^$\aleph_0 > \aleph_0$ and $\aleph_0$-1 is problematic.
By the new notion 3^@ > 2^@ > @ > @-1 etc.

This new approach to the non-finite is not counter intuitive as the Cantorean transfinite universe, because it clearly separates between the continuum (which is **not less** than a non-local urelement) and the discrete (which is **no more** than a set of finitely/infinitely many elements). From a Cantorean point of view, cardinals are commutative ($1+\aleph_0 = \aleph_0+1$) and ordinals are not ($1+\omega \neq \omega+1$).

By using the new notion of the non-finite, both cardinals and ordinals are commutative because of the inherent incompleteness of any non-finite set. In other words, @ is used for both ordered and unordered non-finite sets; moreover, the equality *x*+@ = @+*x* holds in both cases.

## 6. Summary

By observing five-year-old children reasoning about the relations between a point and a line, we defined a meta mathematical framework, which is based on the bridging between the local and the non-local. Armed with bridging, we use symmetry as a measurement tool leading to a new notion of the natural number, based on its internal distinction degree.

By clearly distinguishing between the actual infinity (which is **not less** than a non-local uelement) and the potential infinity (which is **no more** than a set of infinitely many elements), we state that any given non-finite set is an incomplete mathematical object. We think that young children have a natural ability to distinguish between the local and the non local, which influences their ability to understand the Cantorean non-finite.

We believe that further research into various degrees of the number's distinction (measured by symmetry and based on bridging between locality and non locality) is the right way to fulfill Hilbert's organic paradigm of the mathematical language.

Finally, research by Dr. Linda Kreger Silverman over the last two decades demonstrates that there are two kinds of learners: Auditory-Sequential Learners (ASL) and Visual-Spatial-Learners (VSL). Complementary mathematics is a model that bridges between ASL and VSL.

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