Hilbert's Sixth Problem
Proposal for Direction to Solution

Moshe Klein

Gan Adam L.t.d p.box 5024 Tivone Israel 36087
Gan_adam@netvision.net.il

Abstract. This paper extends the known concept of partitions in Number Theory. Inspired by Quantum Mechanics, we define a quantum variant of the natural numbers. We see every natural number as a superposition of its partitions. We take this research one step further and go beyond the partitions by using of recursion. The result that we have is a specific state out of all possible states.

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1. Introduction

Out of the well-known list of 23 unresolved problems in Mathematics presented by David Hilbert [1] during his famous lecture at Paris in 1900, several problems remain unsolved, numbered 6, 8, 12, and 16. The sixth problem was developing new mathematical means for axiomatization of physics (mathematical treatment of the axioms of physics). Hilbert focused in the course of this problem formulating, on the following subjects: Probability, Mechanics and Geometry. First solution to the problem of mechanics was achieved 5 years later. The mechanics changed entirely in 1905. The special theory of relativity which was developed by Albert Einstein [2] fundamentally changed mechanics. In regards to geometry, with the publishing of the general theory of relativity by Einstein in 1915, non-Euclidean geometry was used to describe the curvature of space under the influence of gravitation. The probability theory was put on an axiomatic basis in 1932 by Kolmogorov. It expanded during the raging times of discussion on the actual implication of probability in quantum theory.

Today, 110 years later, many voices are abruptly rising in the scientific community calling for possibility of paradigm change in mathematics that supports the already known phenomena in science. László Lovász, ,wrote in the article 'One Mathematics' [3] on the need for paradigm change and discovering the unity of mathematics by finding a satisfying bridge between continuous mathematics and discrete mathematics. Andrei Khrennikov [4] argues in his work on the need to develop non-Euclidean concept of probability versus the axioms developed by Kolmogorov, which did not address the probability of quantum theory. Michael Atia [5] wrote in a book published following a conference on 'The unity of mathematics' on the signs indicating that the following developments of geometry and mathematics would result from a more profound understanding of physics. Edward Nelson [6] wrote an article on the signs of contemporary mathematics fall. Doron Zeilberger [7] wrote on the shock in which contemporary mathematics is situated.

This article suggests reexamining the relationship between mathematics and physics based on innovative physical principles already established in experiments. The novelty of this article is in its indicating on implementing principles in physics of the quantum theory on the world of numbers. The quantum theory yielded two new principles to science; the principle of superposition and the principle of measurement. In accordance with the superposition principle, each particle, for instance an electron, can be in several places at once. In accordance with the principle of measurement, the measuring of the particle generates a collapse of the wave function for one specific value from a variety of possible measuring results. Using these two principles, quantum algorithms as Shor's algorithm for integer factorization, were developed.

During the development of the quantum theory, attempts were made to develop corresponding mathematical theories to coincide with the quantum theory. An example of this is the 'Quantum logic' theory developed by Von Neumann and Birkhoff. Their main idea was to determine a new set of logical axioms that do not fulfill the distributive law. An additional influence of quantum theory on mathematics was the development of quantum groups that are non-commutative Hopf algebra structures. It should be noted that these significant theories still maintain the deductive logical structure of mathematics.

We ask whether the innovative ideas of quantum theory can be imported directly into the heart of mathematics. We argue that this is indeed possible. We demonstrate this in practice using the idea of number measuring. The number is superposition of its possible measurement values as any partition of it is a possibility of the measuring result.

The birth of special relativity arose as the answer to a very simple question: "How we can measure a length of a body?" The answer of A. Einstein was found by measuring how much time it takes light to cross from one side of a body to the other. This simple observation brings the use of Lorenz transformation of time and space.

In this paper we will ask an analogous question in Mathematics:

"How can we measure a Number?"

Following the discovery of non-rational numbers in the era of Pythagoras, the Greek mathematician Eudoxus determined the concept of equality in mathematics =. Edward Nelson wrote an article entitled 'Confession of Apostate Mathematician' on the significant contribution of the Greek mathematician Eudoxus to the development of mathematics. Introducing the concept of equality was undoubtedly the decisive step in the establishing numbers existence. However, the philosophical establishing of the number concept is controversial. The realistic approach considers numbers an actual entity independent to man, as objects in the world. The intuitive approach argues that numbers are product of the mathematician's imagination. On the other hand, the formal approach ascribes importance to the symbolic description of the number in its mere writing.

The question is whether and how can the different perceptions of the number concept be bridged? We propose a concept bridging the number concept using the two principles of quantum theory; the principle of superposition and the principle of measurement.

We consider the partition of a number somewhat of a number measurement result. Below is representation of the various partition of the number 4. Each partition is represented by numbers and parentheses.

4=1+1+1+1= ()()()()
4=1+1+2= ()()(())
4=2+2= (())(())
4=1+3= ()((()))
4=4= (((())))

Subsequently we expand the idea one-step further by using recursion. Finally, we omit symmetrically repeating situations. The final result received is the quantum form of a number.

The significance of the new perception is in demonstrating the mathematician's involvement in the mathematical symbols system he creates. In our opinion, there is no possibility of genuinely understanding why mathematics operates in a world of no consensus regarding the principle that mathematicians themselves are inherent part of the world's phenomena. This organic understanding requires developing a new mathematical language in analogy to the developing of an integral and differential mathematics calculus, in which the pace or process of transformation is an important factor in the true understanding of the function.

The process of mathematics creating is an internal recursion that in some cases is endless. Steven Rosen [8] and Diego Rapoport [9] wrote on the importance of executing radical recursion. It is possible to surmounts the Cartesian Cut through Self-reference. The real way to realize it, to their understanding, is by the model of Möbius strip or Klein bottle. Such imaginary reality allows creating a bridge between an inner and outer language.

In recent work D. Shadmi , Moshe Klein [10, 11] present the concept of a number as bridging between the continuum and discreetness. This paper applies this attitude as well as brings inspiration from Quantum Mechanics to our concept of measuring Integers. We believe that the answer to this question might help to solve some of the challenges in physics today. Such as more understanding of Non locality as described in EPR, and perhaps formulate new foundations and interpretations to probability as challenged by A. Khrennikov.

In receiving these ideas, we pave the way to the solution of Hilbert's sixth problem on understanding the nature of the relationship between mathematics and physics. The process of number measuring considers uncertainty an inherent part of the language of mathematics. This way, transformation on the concept of probability is indeed formed. We do not intend to present a final solution to Hilbert's sixth problem. Our perception of reality is based on the concept of interaction. We propose a way in which one can operate in along with appropriate theories to promote a joint solution to this important problem.

2. Partitions

At a first glance, it appears that there is no difference whether one writes 1+1=2 or 2=1+1. But a deeper observation shows that there is a significant difference between these expressions: The first is a simple addition exercise between two numbers, while the latter describes a possible representation of the integral 2 as a sum of two integers. The Trivial Representation 2=2 serves another representation for the number 2. In general, a partition of a natural number is a way to write it as a sum of positive integers. In elementary number theory, a partition is only made up of the numbers being summed and not from the order of the summation. For instance, all of the partitions of 5 are listed below:

5=1+1+1+1+1
5=1+1+1+2
5=1+1+3
5=1+2+2
5=2+3
5=1+4
5=5

We take as convention that the smaller number appears first. We define the set that contains these 7 partitions, Par (5) (called the "partition set of 5"), and we call each of these elements an element of Par (5). E.g. {2+3} and {1+4} are elements of Par (5). For every positive integer n, the Partition Function, p(n), gives the number of different ways to partition a number, or in other words, the number of elements in Par(N). For instance p(5)=7,p(4)=5,p(1)=1. It is interesting to mention that, there aren't any explicit formulas that calculate the Partition Function for a given n.

We define a probability distribution over the partition. E.g. for Par(5) we may define Prob({5})=p and Prob({1+4})=1-p. Inspired by quantum mechanics, we define a quantum variant of the natural numbers: For this, we use the bracket notation. For instance: $|\varphi_1>=|5>$ with probability p and $|\varphi_2>=|1+4>$ with probability 1-p is an example of a probability distribution, having the same meaning as the probability distribution mentioned above. Due to the use of quantum states, one can go beyond probability distributions. As an example, $|\psi>=\cos(\phi)|5>+\sin(\phi)|1+4>$ is an example of a superposition of the two (orthogonal) states mentioned above. Finally, one can also define a density matrix (a probability distribution over quantum states) such as: $q|\psi><\psi|+(1-q)|1+1+1+1+1><1+1+1+1+1|$.

3. Indistinguishable Elements

On the table there are five balls. Suppose we are able to distinguish them using some property (for example, color). We therefore sort the balls according to their color. We also take into consideration the possibility that we will find that some balls are identical according to this property. As an example, we might have 2 balls that look the same (they have the same color) and the other three look the same as well, yet different from the first two. We will denote it this way: (x, x, y, y, y) . If we do not care about the order of the balls, we observe that this situation corresponds to the partition element {2+3} of Par(5).

Let us look at basic examples: If there is only one element, it is distinguished (x). This corresponds to the only partition for 1, namely 1=1, which we denote as {1}; If there are two elements there are two possibilities: Either the state (x, y) or the state (x ,x). These correspond to 2=1+1, or 2=2 respectively. When we have 3 elements we have 3 different states: (x, y, z), (x, y, y) and (x, x, x), corresponding to {1+1+1},{1+2}, and {3} respectively.

In quantum theory, indistinguishable particles play an important role, as they have special statistics. The partition {2+3} could then correspond to two indistinguishable particles in one mode and three indistinguishable particles in another mode.

4. Recursion

We now take this research process one step further and try to go beyond partitions:
On the table there are still five balls. Now, each ball has a vector of properties. (color, mass, size). Let us suppose too that we have a particular order in which we can check their properties: Say, first color, then mass and then size. We want to sort the balls according to their different properties: At first, we sort them according to color: Suppose that judging only by color we have the same situation we had before –
(x, x, y, y, y). Now, if we weigh every ball, we might find that some of the balls that were indistinguishable before are now distinguishable. For instance, suppose we find the two x-balls to have different mass, and also, one of the three y-balls is different from the other two. This can be denoted as $((x_1, x_2),(y_1, y_2, y_2))$. This corresponds to something like a recursion over the partition: 5=(1+1)+(1+2). Using our notations { }, we can denote the split of elements into sub-elements as follows {2+3} changes to {{1+1}+{1+2}}. Now we have two balls that are undistinguishable and we still have properties that aren't checked say, size. After checking size we will have one of two options, and this depends on what happens to the pair of y2's. Either {{1+1}+{1+{2}}} or {{1+1}+{1+{1+1}. As long as there are more properties to test, we may still find that finally all balls are distinguishable.

Let us try to stick to the recursion, without looking at a specific number of different properties. At each level of the recursion, we give each number the chance to get split. For instance, 2=2, and 2=1+1 can be found after the first level of recursion.

Since the next level can only generate 1+1, which we already have, we stop here. Otherwise, we will have an infinite recursion (checking the number 2 again and again). For the number 3, this means that the elements {1+1+1} and {3} (obtained at the first level) do not need to be investigated further, and only {1+2} can either lead to {1+{1+1}} or to {1+{2}} at the second level of recursion. There is no need for a third level of recursion here. The general rule of thumb is that item {k} never requires another recursion. In contrast, {1+k} does have recursion over the number k, and {m+k} has recursion over m and over k. By the end of the recursion, items such as {1+k} or 2+k} will not appear, and each element will contain 1's and items in parenthesis: {k}, {m} etc.

We list all possible states we can achieve from 1-3:

1:=(x) which corresponds to {1}

2:=(x ,y);(x ,x), which correspond to {1+1}; {2}.

3:=(x, y, z); $(x,(y_1,y_2))$;(x, (y, y)) ;(x, x, x), corresponde to
{1+1+1}; {1+{1+1}};{1+{2}};{3}.

In a collection of 4 items we have the following possible states:

4:=4	(x ,x ,x ,x)
4:=1+3	(x,(y ,y ,y)); $(x,(y_1,(y_2,y_2)))$; $(x,(y_1,(y_2,y_3)))$; $(x,(y_1,y_2,y_3))$
4:=2+2	((x ,x),(y ,y)); $((x_1,x_2),(y_1,y_2))$;
	$((x_1,x_2),(y,y))$; $((x,x),(y_1,y_2))$
4:=1+1+2	$((x,y,(z_1,z_2))$; $(x,y,(z_1,z_2))$
4:=1+1+1+1	(x ,y ,z ,w)

One can see that for the number 4 we have 12 states.

5. Removal Of Redundancy

One can notice that the two states $((x_1,x_2),(y,y))$; $((x,x),(y_1,y_2))$ correspond both to 4={{1+1}+{2}}. However, we clarified that order does not matter, so we must conclude that these two elements are identical. Thus, omit one of them, while we keep as convention that the smaller number appears first. The first bolded state corresponds to 4={{1+1}+{2}} and the second to 4={{2}+{1+1}}, so we omit the second case. We are left with 11 states. In general in all partitions that have repeating numbers different than 1 (like 6=3+3 or 5=2+2+1) we can omit certain "symmetric" states.

The process of distinguishing between the balls may correspond to a process of measuring an integer: We are given a natural number N (or its quantum variant) and we measure one of those final states. The result that we have is a specific state out of all possible states. For instance for N=4 we start with 4 balls and after the investigation we get one of the 11 states mentioned above.

6. Conclusion

The different quantum states that we reach in the final step of the recursion represent different ways in which the observer can influence the world of Numbers. We believe that these ideas can be applied to other concepts in Mathematics.

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