George Spencer-Brown

Uncolorable Trivalent Graphs

Part 0. Prescript

The text that follows is an extended write-up of the Ross Ashby Memorial Lecture of the International Federation for Systems Research, delivered to a plenary session of the Thirteenth European Meeting on Cybernetics and Systems Research at the University of Vienna on 1996 April 10th. For ease of reading I have retained the informal style of the lecture theater, but without sacrificing accuracy and rigor in my theorems and proofs.

Isaacs (1) detailed a way to make uncolorable 3-graphs. Part I of this communication shows that it is, in principle, the only way, and that all such graphs that can define maps must be nonplanar: thus proving the 4-color map theorem by exclusion. Part II shows that the method is ineffectual for closed chains of units with a linkage of less or more than 3, so proving that Issacs found all uncolorable 3-graphs. The associated proof of the map theorem in no way depends on this stronger result, but on the fact that chains with a triple or greater linkage, whether colorable or not, cannot be planar.

Part III simplifies the Isaacs dot product, proves that his Q class has no more members, and relates the 4-color proof in Part I to the proofs in Appendix 5 of (2).

Part I. A short proof of the map theorem

The Isaacs paper of 1975 was a revelation. For the first time since the publication of Petersen's graph in 1891, it brought new ideas to the study of uncolorable 3-graphs. In a profession where everybody writes, but hardly anyone reads, the work did not receive the attention it deserved. Most mathematicians of my acquaintance have not seen it, and those who have did not appear to understand its significance. Not one has developed the Isaacs methods or re-worked any of his proofs.

Since the present paper is a sequel to it, and answers all the questions posed there, the reader may find it useful to have a copy of reference (1) at hand while (s)he studies what I write here. In the interests of brevity I do not explain or re-prove here what is explained or proved correctly there, but I do simplify and develop the Isaacs methods and correct or augment what I consider to be mistakes or inadequacies in his proofs. It will also be helpful to have reference (2) at hand, where I explain in more detail what is mentioned only cursorily here. Both of these references are essential to a proper understanding of what I write here, and no other preliminary reading is necessary.

In Part I of this communication I shall offer a prediction that Isaacs found all the uncolorable 3-graphs. If my prediction were a proof, it would of course prove the four-color theorem, since all the Isaacs uncolorables that can define maps are nonplanar.

I will then proceed to show, irrespective of whether my prediction is correct or not, that Isaacs supplied a simple method of proving the map theorem that he failed to notice. That he overlooked it is not surprising, since I did not at first see it myself, and I had already published the appropriate mathematics more than a decade earlier.

It has always been difficult to describe the essentials of any graph without confusing them with how we happened to draw it on paper, and Isaacs was certainly not blameless in this respect. The reader will also see that I prefer a neutral terminology. Where Isaacs uses the term ‘edge’, I call it a line, lead, or link. Where he writes ‘vertex’ (a point where lines meet), I call it simply a node. My usage of other terms will become apparent as we proceed.

We first establish that a connected set of four nodes is the natural atomic unit that can be linked with other such units in chains. A single node won't do, because it has free valencies that allow the chain to branch. A node with three links is a unit, but requires nodes at the other ends of the links to connect them to other such units. So the set of 4 nodes is the first natural unit, that can be placed in a reentrant chain of such units, each with six loose ends, which Isaacs called ‘pendants’, three of entry and three of exit round the circuit.

Any number of nodes greater than four, either does not form a natural unit of paradoxical reentry (to be explained below), or can be factored into natural minimal units of four. To predict this we have only to show that we cannot make a 6-noded atomic unit that can be used in a paradoxical chain. We do not have to put more than three such units together to make a minimal chain, and the trivial middle 3-ring can be collapsed into a single node, as in the petersen. If we cannot make an uncolorable (and irreducible) superpetersen of 16 nodes we are done, and of course we cannot.

No new principle is involved in making chains of still larger units, and this is the basis of my prediction.

We now define paradoxical reentry. I explain in Chapter 11 of (2) that an odd number of units can be linked together in a closed chain so that whatever value (which may, for example, be a color) is assigned to a link x, the chain connexion round the circuit will determine that x cannot have that value.

When this happens we call the circuit paradoxical, and each successive unit in the chain alternately paradoxifies and deparadoxifies the circuit. Hence the requirement of an odd number of units.

In (2) the units are single markers, and the links are the two leads of connexion, one of entry and one of exit round the putative paradoxical circuit.

In colored 3-graphs, as we have established, the units are sets of four nodes with six leads of connexion, three of entry and three of exit round the putative paradoxical circuit. As before, each successive unit will alternately paradoxify and deparadoxify the circuit, so that the whole uncolorable graph will require an odd number of such units, totaling 4, 12, 20, etc nodes, provided we can find a set of connexions with the required paradoxical property.

J_1 is the degenerate case, not noted by Isaacs, of the primary unit of 4 nodes connected to itself. The resultant graph is singly-connected and planar, and of course uncolorable because of the single connexion. It cannot define a map. J_3 is the first proper case, 12-noded but reducible to the petersen P by eliminating the trivial 3-ring. J_5 is the first nontrivial case, with 20 nodes, whereafter no new principle is introduced.

Isaacs showed (and I will show more simply in Part III) that other uncolorables can be made as extensions, elisions, or trivializations of these. We can show by exhaustion of cases that the way he adopted is the only way of connecting his units paradoxically. More importantly, it can be seen that no other way, other than some such set of connexions, can possibly make a closed circuit of such units.

Although it is evident that there can be no other way of producing uncolorable 3-graphs, other than to make them paradoxically reentrant, it is completely irrelevant whether Isaacs found all of them or not. For just suppose there happens to be a larger unit, say of 6 nodes, that Isaacs missed, and that can be linked in an odd chain of such units to make a paradoxical graph. Three such units are enough to make the first proper case, and we can amalgamate three nodes in one to make a superpetersen S, say. Suppose each unit of 6 nodes has the common central node shared, as in the original petersen P.

The pendants *a, b, c* must be connected in some order to pendants *p, q, r* leading to *a', b', c'* connected in some order to *p', q', r'* leading to *a'', b'', c''* connected in some order to *p'', q'', r''* leading to *a, b, c* again.

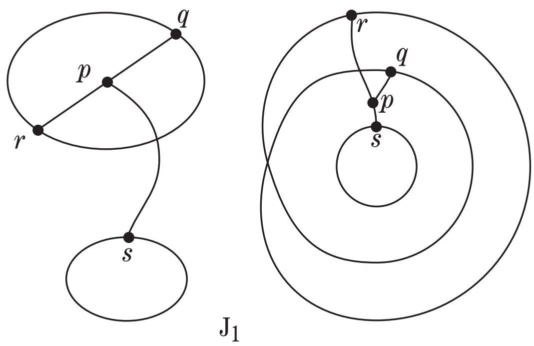
However many nodes in the unit, and however many units in the chain, this way of connecting the units has to be nonplanar*, and it is evident that no other way will make a circuit at all. It is completely irrelevant whether any of the graphs so constructed is uncolorable, or whether none of them is, since they must all be nonplanar.

Thus no uncolorable 3-graph, of any that can make a map, can possibly exist in the plane.

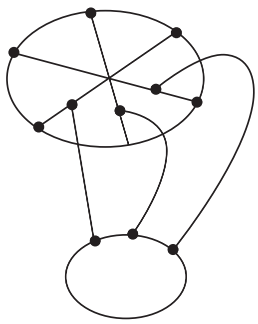
I do in fact prove this in a more traditional way in Appendix 5 of (2), 1997 (not in earlier editions). That proof is more instructive. This proof is shorter and pleasing because of its unexpected nature, so obvious once it is pointed out, so invisible before.

[London 21 06 1997]

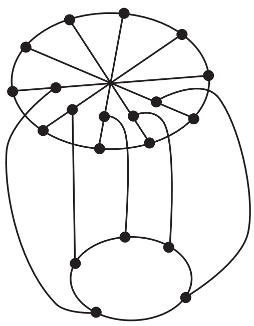
* The proof of this, which Isaacs never bothered to make in general, hence his rather fanciful notion that the unicorn might after all be planar, is simply that they can all be seen to contain J_2 with the 2-ring collapsed, or figure 1.1 with each of the three 3-rings collapsed, both turning out to be the utilities graph U. The way I draw it, the unicorn can be seen to contain U in all sorts of different ways. But the beauty of including 1.1 is to see that any chain we can construct, with a unit-count of three or more, and a linkage between units of three or more, trivial or nontrivial, colorable or not, must contain U and is therefore nonplanar.



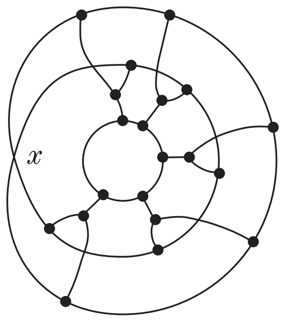
J_1



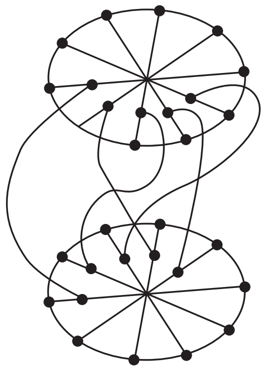
J_3 (elimination of the trivial 3-ring reduces it to P)



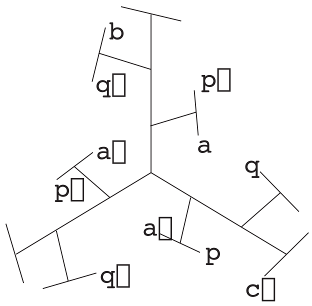
J_5 (is an extension of J_3, and J_7 etc are extensions of J_5)



J_5 as Isaacs dissected it. He misleadingly called the chiasma at x a ‘crossover’. As can be seen from my symmetrical drawing, the chiasma is an artefact with no real existence in the graph.

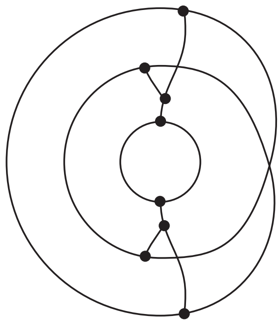


The Unicorn (misnamed by Isaacs “Double Star” as a result of more artefacts in his drawing. My name for it is a contraction of Unique Ornament). We see it is an elision of two J_5’s.

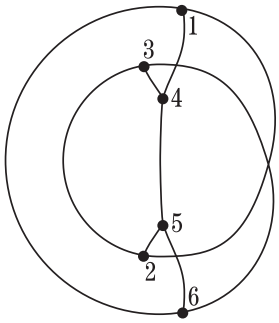


Putative “Superpetersen” S

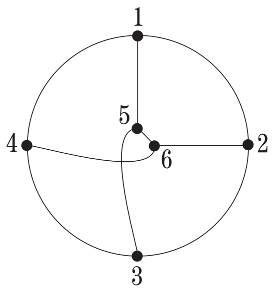
A simple pictorial proof that all the chain graphs we have considered are nonplanar.



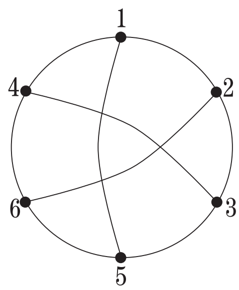
J_2



J_2 with the 2-ring collapsed



The same



The same again – the utilities graph U, the simplest nonplanar graph.

All the larger J's are evidently adjuncts to U, and all chains of larger units are evidently adjuncts to J's. For a completion of this proof see figures 1, 3 and captions.

Part II. Criteria of uncolorability

We are “lucky” in having a reentrant chain of no more than 3 identical units to check out all our examples. For if a chain cannot make a nontrivial uncolorable with three units, it cannot make one with more.

Let us recapitulate what we know. By (2), map-theorem 12 (p 159), we see we can call all graphs less than 5-connected trivial, simply because if they are uncolorable they cannot be smallest. All trivialities in such graphs are 'reducible'. That is to say, they can be eliminated from the graph to leave a ≥ 5-connected graph that is also uncolorable. The prime case is J_3, which by elimination of the trivial 3-connexion becomes the petersen.*

Call graphs that are reducible to the same graph *equivalent*. We may write a sign = between equivalent graphs.

We review next what we know of closed chain circuits, and how to make them. In (2) Chapter 11, Equations of the second degree, the chain links are 2-valued.

W — a | b | a | b — E

If the exit E is connected to the entry W, we have a paradoxical chain, consisting of units of distinction (markers) and links between them, such that the value of the link at E will always contradict the value of the link at W.**

In (2) Appendix 5, Two proofs of the four-colour map theorem, we see that the chain must be 3-valued.

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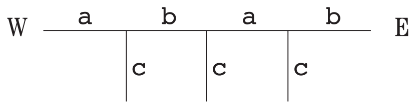
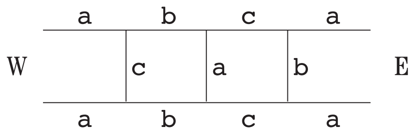
* The term ‘reducible’ and its relation ‘trivial’ were first used in the context of ‘if’: ‘if’ the graph were uncolorable, it could not be smallest. But the ideas are more fully expressed in terms of information theory, when it is seen they must apply to graphs (and other algebraic equations) in general irrespective of “uncolorability“. For example we can take a 3-node and expand it with any graphical configuration whatever, without in any way altering the mathematical properties of the rest of the graph. All 2- and 3-connected zones in the graph are effectively “black holes“, or what I call placental zones, since their links with the rest of the graphic universe are insufficient to allow the transfer of any information either way. Less obviously, the same considerations apply to 4-connexions, as was proved for planar graphs by Birkhoff in 1913, and for all graphs by me in 1979.

** Consider the line a b . . . as a great circle round a major circumference of an anchor-ring, and the markers between indicated values as little circles round minor circumferences. Now a, b can be seen as a paradoxical 2-coloration of the map whose borders are the markers. Such a map can be considered as a second-degree equation with imaginary roots. By my chromatic formula for nodeless maps (given in (2) p 183, theorem 27) we see that the torus-surface, of connectivity 3, is the least-connected surface in which such an equation can appear. Here the addition of a third color, c, substituted for one of the a’ s or b’ s, will block the paradox. My 4-color map theorem can therefore be seen to state and prove that in the plane, all minimal equations for maps must have real roots. Thus my Chapter 11 of (2), first published as lecture-notes in 1960, already comprised a published proof of the map theorem, since it implies my theorem 7 in Appendix 5 (p 150). The fact was recognized by the few friends and colleagues who understood it, including and on record J C P Miller, Bertrand Russell, D J Spencer-Brown, and later the renowned graph theorist G A Dirac. Dirac was adamant, as were Miller and I, that Haken, Appel, and Koch had not proved the theorem, and that their claim was no more than a mish-mash of half-digested ideas and unread papers of previous investigators, together with some irrelevant computer printouts containing gross computational errors.



If all of an odd set of connecting markers could have the same value, say c, then the links between them could be paradoxically valued if the circuit were closed by connecting E to W. In fact no odd set of connecting markers can all have the same value, so the values a, b, c (which we can denote by different colors) must distribute themselves in other ways. The ladder graph



is not paradoxical, therefore colorable, besides being trivial.

To create a paradox of such triple values we must be more ingenious. And Isaacs supplied the ingenuity. His proof of the uncolorability of his J class is in fact a recipe for making any of its members, which as it happens turns out to be the only way. See figures 1, 2.

Since in the first proper case (a chain of 3 units) the graphs are manageably small, the circuits can be alternatively proved nonparadoxical by finding a coloring, and proved paradoxical by examining all cases and finding no coloring. Extrapolation from either case is permissible, though of course the Isascs argument* in the uncolorable cases renders it unnecessary. But since a general proposition can be falsified by finding a countercase, it is a correct method, and one we shall employ here, to show that the Isaacs argument will fail with units having four or more loose ends to a unit, and thus four or more connecting links between successive units.

From here on we are counting the number of loose ends as the number of free projections from the unit nucleus without the end-nodes, i.e. the number that determines the magnitude of the linkage between units. Thus the unit in the Isaacs uncolorables is now considered to be one node with three loose ends.

In the first edition (1969) of (2), I had already shown that we cannot make a 2-colored map paradoxical unless we draw it on a nonplanar surface. Isaacs produced, without recognizing it, an exactly similar phenomenon with a 3-colored graph. But granted he had found one way to make his graphs paradoxical, it still remains to show that there is no other way.

* But see Note 1 at the end of this section. I first considered the Isaacs argument to be a brilliant proof of uncolorability. It was only after trying to extend it to linkages > 3, that I discovered his argument did not actually exist, i.e. was not a proof at all!

We have established that a reentrant chain is necessary, and that three units are enough. The only possible alternatives are to make the units larger (i.e. to allow them more loose ends) and/or to connect them a different way.

The only other nontrivial way to connect the units is by a reentrant spiral traversing all the loose ends of the units. We need only to color two consecutive cases, since the only difference in the coloring algorithms required is when the minimal number of internal nodes in the units is odd or even. Figures 3.1, 3.2 are two such cases. The repetitive coloring algorithms are so simple that I can safely trust the reader to find them him- or herself.

We now see the advantage of using a 3-unit chain. For if we find it is nonparadoxical, then no longer chain of similar units can be paradoxical. And if we are to avoid triviality, there can be but one other way to connect the loose ends of the units: notably with just two rings, the smaller of 3 links that can be detrivialized by collapsing it, as in the petersen, and the larger, of at least twice as many links, spirally connecting (= connecting in any order, it makes no difference) all the other loose ends of the units. See figures 3.3, 3.4. Again we can look to find two successive cases colorable, this time with respectively 2 and 3 internal nodes in the units, following which we know we can color all chains of larger such units.

We eliminated the trivial 3-ring by collapsing it into a node. We see we cannot break the larger ring without making the graph trivial (3-connected), so there is no other irreducible way of connecting the units. Colorings are a little harder to find in these cases, but I think I can still safely leave them to the reader. Having colored two successive cases, we now know we can color all chains of such units with more loose ends, since the variety of possible colorings can only increase (and can never diminish) with the size of the linkage between units.

There is now but one remaining possibility. Suppose some of the internal nodes in the units are reconnected to each other, instead of supplying loose ends for the spiral connexion of the chain links? A moment's reflection shows that it makes no difference, except possibly to make the coloring easier. By the simplest other-than-minimal internal connexion, connecting all loose ends internally to a ring, we see that color-variety in the loose ends can be increased* but not diminished, rendering the chain links sometimes easier to color, never harder. By making enough internal connexions, we can even make the graph planar, but it will then be trivial and easy to color – the equivalent of a ladder graph.

We can thus make internal connexions any way we please. Our only concern is with the loose ends connected in some order to the common linkage of the chain. The ring-

* An even number of loose ends can all be assigned the same color, rather than the two colors required by the tree-connexion, allowing an obvious coloring algorithm for the outer ring in these cases.

coloring problem for k loose ends with any larger number of internal connexions is either exactly similar to or easier than the problem for k loose ends minimally internally connected, and the case of $k = 3$ is the only case where no coloring is possible.

The general proof of this is to see that no form of the Isaacs argument (or rather, of my augmentation of it in Note 1 below) can apply to linkages greater than 3, where the variety offered by more links than colors makes it impossible to compel a paradoxical connexion. And since the variety increases with the linkage, there is in fact no need to look for colorings beyond the 4-link stage to know that, if all nontrivial linkings at this stage are colorable (the doubtful reader should try them all – see Note 2 below), then any chain with a greater nontrivial linkage must also be colorable.

Since I have covered all possible cases, my prediction in Part I of this communication, that Isaacs found all uncolorable 3-graphs, is now proved. And in proving it, it will be seen that I have proved the four-color map theorem twice over.

It should be observed that this associated proof of the map theorem also stands quite independently of whether my claim to the stronger proposition is deemed to be true or not, since it can be seen to rest entirely on the much more obvious proposition that chains with a triple or greater linkage cannot be planar.*

[*London 0351 27 06 1997*]

* My proof here is so simple and obvious that it surprises even graduate mathematicians into thinking it cannot have happened. One of my PhD students was so disturbed by it he asked me again and again, 'How can you be sure all uncolorables can be constructed this way?' I told him, the question is better phrased, 'Could any uncolorable be constructed some other way?' Now the answer is obviously No, since if the graph does not turn back on itself to compel the contradiction of some color we have already applied to one of the links, then we are certain to be able to color it.

Note 1

Although his "argument" is a picturesque way of seeing the contradiction in the paradox, and very helpful to the reader, it is not strictly correct. Uncolorability, like primality, with which it has much in common, is ensured only by exhausting all other possibilities. Isaacs uses clockwise and counterclockwise, and these have meaning only as seen from one side of the plane. Witness Alice through the looking glass. But Isaacs is not in the plane, so in his space clockwise and counterclockwise are indistinguishable. It is only by squashing his graphs into the plane (where they won't go) in a particular way, that enables him to do the conjuring trick that persuades us that this is how they should look. He then proceeds to draw them another way, extremely confusing to the reader, and evidently to himself, by insisting that the nonexistent "crossover" should be drawn somehow differently in what is in fact a perfectly radially symmetrical graph.

But although his argument is fallacious, his conclusion is nevertheless correct. What he has really done is what I do, tried every other way of connecting his graphs and discovered this is the only way that works. He has then extrapolated, as I do, using a multiplex form of mathematical induction, proving we can get parodoxicity in chains of all odd numbers (n) of units provided we can demonstrate it for some unit with $n = 3$.

I have here employed a similar multiplex induction for an opposite purpose, to prove that if we cannot get paradoxicity in any nontrivial case of a specified linkage k , when $k > 3$, then we cannot get it in any nontrivial case of $k + 1$. I then complete the induction by demonstrating, or pointing out how the reader can demonstrate, the absence of paradoxicity in all nontrivial cases of $k = 4$.

Note 2

Referring to figure 3.3, (s)he will see there are only two such cases. We see 3 loose ends near the central node, call them N, and 6 distant, say D. We can connect them DDDDDDNNN, as I have done, or DDDDNDNDN. The coloring algorithm is to find the correct colors for the loose ends, and it is identical for both cases.

Note 3

All the graphs in figure 3 are hamiltonian, that is, have a circuit that crosses each node just once. The coloring is easy in these cases, color all the links in the circuit alternately with two of the colors, and now all remaining links can be colored with the third.

So forget about general induction for the moment and look for coloring algorithms. Such an algorithm could be a rule that lays out the pattern for the hamilton circuit in a given class of cases. This is easy to find in extensions of 3.1, 3.2, where the algorithm is the same for both, and scarcely more difficult in the cases beginning with 3.3, where the number of internal nodes is even. 3.4 I found harder, and was not sure I had the algorithm until I had done all cases of 3, 5, 7 internal nodes. Then the repetitive pattern became obvious. My general method of instruction, having pointed the way, is to leave as much as possible for the reader to rediscover.

In any case, whether you find the algorithms yourself (as you really should) or are willing to take my word for it that they exist, we shall between us have made an alternative proof of my theorem (that Isaacs found all the uncolorables) by finding a coloring algorithm for every other possible class of cases. Either method of proof is good, provided it convinces you that all possible nontrivial linkings, other than those that Isaacs found, must be colorable.

There is yet a third, more sophisticated, proof that all alternatives to the Isaacs graphs are colorable. It stems from the self-evident proposition that the reduced form of every uncolorable must be radially symmetrical. (If you do not yet see this as self-evident, do not persist with this proof.) The evident asymmetry of all the graphs whose least members are in figure 3, indicates that they must all be colorable.

We have thus without too much difficulty made three independent proofs of the same theorem. They are

1. Paradoxicity is definitively a choiceless phenomenon, and any nontrivial linking of a linkage greater than the number of colors must offer a choice, therefore no paradox.

2. We can define coloring algorithms for all nontrivial linkings of linkages greater than 3.

3. All nontrivial graphs of minimal unit 3-unit chains with linkages greater than 3 are evidently asymmetrical, therefore colorable.

I suspect the reader at this point will feel shocked by the finality of these proofs and the apparent ease with which they have been achieved. Might there not be some other way to make a paradoxical 3-graph? Well, we have done everything we can with the linkage so far, except take it *through* a chain unit, i.e. through a *link* in the unit, instead of tangentally through just one external node. But this is expressly forbidden, because it allows us when required to exit from the unit with a link of the same color as the link that entered it, which obviously destroys any chance we ever had of making a paradox. Stated more formally, the seperate strands of the linkage loose their identity in such an arrangement, so paradoxicity becomes impossible to define. And what cannot be defined cannot happen.

Taking some or all of the linkage, in cases of k ≥ 3, through a chain unit is also a way of allowing the graph to be planar. See figure 4. The so-called "difficulty" of the color problem was the failure to see that this, even with just *one* strand of the linkage through just one of the units (and to avoid triviality, we must take at least two strands through), must evidently destroy all possibility of paradoxicity. For decade after decade in the present Century, investigators vied with each other to prove that larger and larger plane maps must all be colorable, yet still "officially" imagined (though none of them really believed it) that they might "suddenly" discover some increasingly large map that would somehow turn out to be a "smallest" plane map that could be paradoxical. But they did not *say* 'paradoxical'. They said 'uncolorable', and this failure to identify the proper mathematical paradigm is what made the whole enterprise so fruitless.

By simply substituting the correct* term 'paradoxical' for 'uncolorable', I have eliminated whatever "difficulty" there ever was to the 4-color problem, and made it easy.

It is equally evident that every well-known unsolved problem of mathematics rests on some such similar misconception, which when rectified will make it as easy as I have made this one. I have in fact shown it to be so in other cases.**

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* Correct because the graph is colorable, but with imaginary colors, not real ones, just as the equation $x^2 + 1 = 0$ is solvable, but with imaginary (i.e. paradoxical) numbers, not real ones. For a more-detailed exegesis see Chapter 11 of (2) and the notes thereto, also p xxii, which is paraphrased from a letter dated 26th September 1967 I wrote explaining it to Russell, and acknowledged by him in a letter to me dated 28th September 1967.

** For example in my proof of the existance of primes between all perfect squares. See Appendix 8.

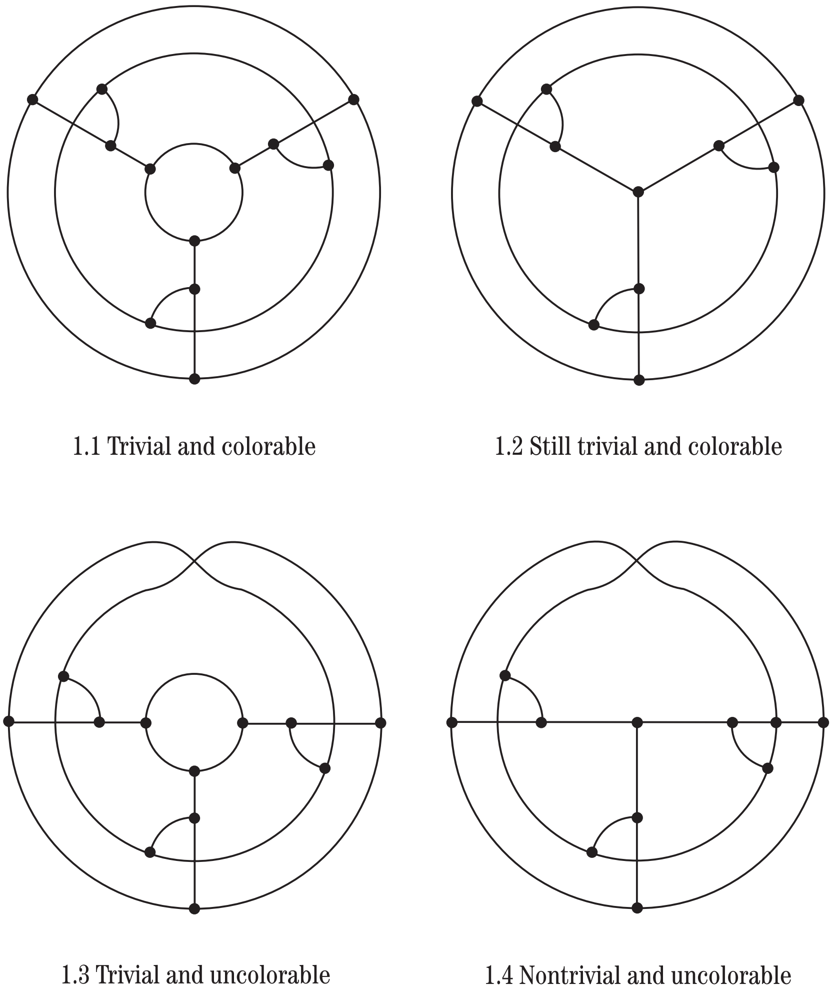
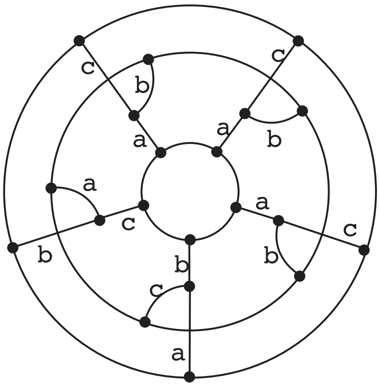
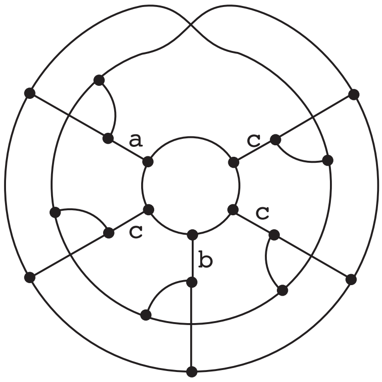


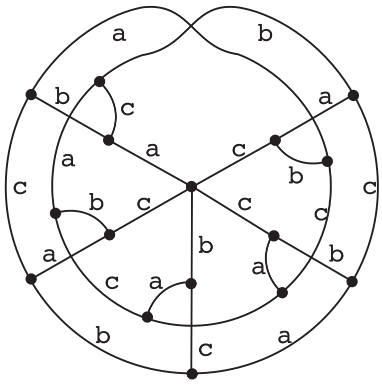
Figure 1. Notice that 1.1 with the 3-rings collapsed is U. This extends my proof that all chains of unit-count and linkage ≥ 3 are nonplanar. Since J_2 and 1.1 are equivalent, the nonplanarity of these chains is already evident without invoking Kuratowski's theorem. For the final case see figure 3.



2.1 Nontrivial and colorable

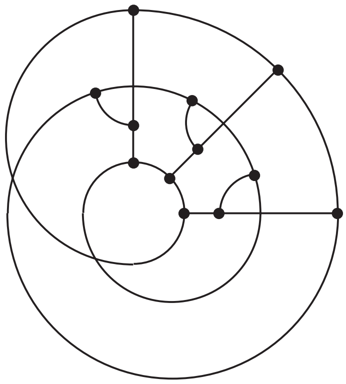


2.2 Nontrivial and uncolorable

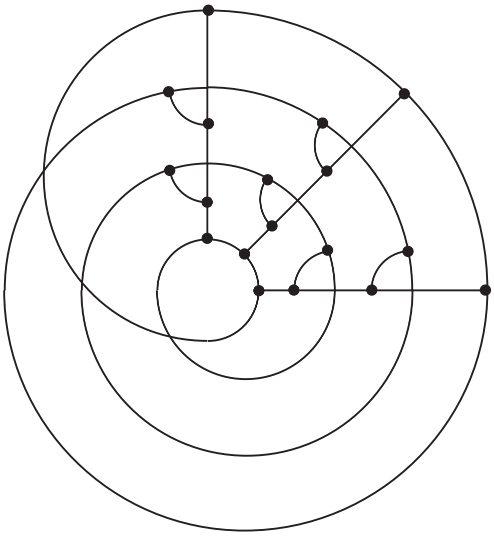


2.3 Nonstandard and colorable

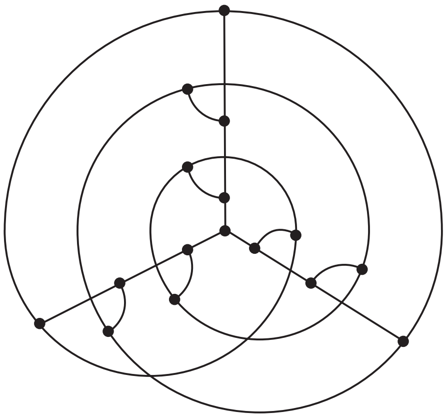
Figure 2

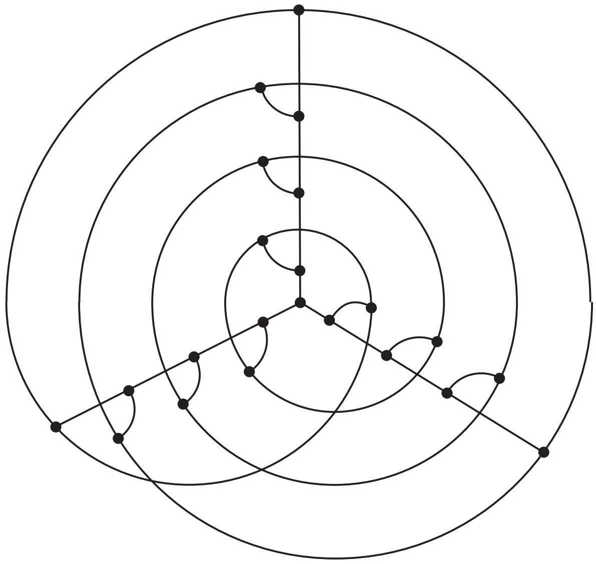


3.1



3.2





3.4

Figure 3. All colorable. As I point out at the end of the text, a coloring of 3.3 in fact implies colorability in 3.4 and all cases with more loose ends. There is a simple way to show that 3.1 and extensions of it are nonplanar, and thus complete my proof of nonplanarity in all cases. Can the reader see what it is? (Answer. Number the three central nodes sequentially 1, 2, 3 and continue the sequence 4, 5, 6 through the next three nodes in the spiral. Remove all links to the spiral except those directly connecting 1 to 4, 2 to 5, 3 to 6. What remains is U.)

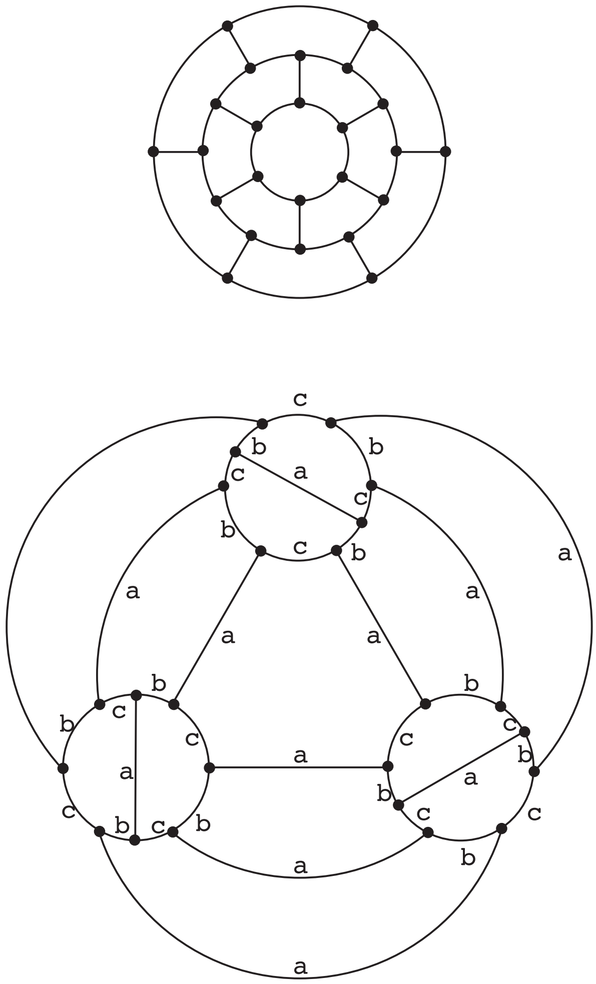


Figure 4. A radially symmetrical graph displayed (i) as a plane map, and (ii) as a reentrant chain graph of linkage 3 with all strands proceeding through the units (i.e. via more than one node in a unit). At least one strand must proceed this way in the plane, proving that all planar graphs are colorable. The reader should experiment. See Note 3.

Part III. Postscript

The uncolorable graphs so far considered (the petersen, the J's, and the unicorn) I call *primary*. Isaacs demonstrated how two or more primaries could be combined to make a *secondary* uncolorable graph. He called it a 'dot product'.

The instructions Isaacs supplied to make his dot products are complicated, so I devised a simpler method, called *nuclear insertion*.

Suppose we have two uncolorables K_1, K_2 that we wish to combine. Any two will do, similar or different, I illustrate with petersens merely to save labor.

1. Select any link in K_1 and expand it from the nodes to make a 2-ring. The original 3-nodes of this link will now have been expanded into 4-nodes.

2. Remove any ring component in K_2, leaving a nucleus with loose ends. Thus if we take out the ring component of a 5-ring, we will be left with a nucleus with 5 loose ends that were connected to the ring.

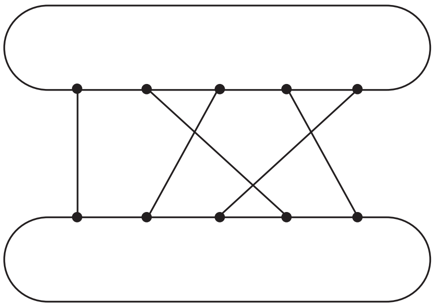
3. Insert the nucleus from K_2 into the expanded link of K_1, taking care to preserve the cyclic order of the loose ends. Thus only one loose end may go, should we wish to put it there, in each of the enlarged nodes of the expanded link of K_1.

4. The resulting nonstandard graph is uncolorable, and we may standardize it as we please. The zones containing the expanded nodes I call *placental zones*, since they are effectively nerveless to the rest of the graph, so it doesn't matter how we connect the links there.* If an odd number of links enter a placental zone, we shall need at most one node to combine them (we may of course use more, since we can put anything we like in here), if an even number, we shall need none.

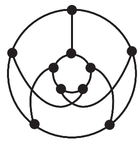
It will be observed that the resulting uncolorable graph can be at most 4-connected, thus trivial. It is possible to do a second insertion in it and make the resulting graph 5-connected, thus masquerading as a nontrivial graph. But it can be seen that, by suitably connecting the placental links, every such graph has a 4-connected equivalent. This solves the Isaacs puzzle (p 232 of (1)) concerning zonalities in his BDS class.**

* Neither Isaacs nor Tutte (who was obviously his mentor and referee) recognized this phenomenon. Tutte, in the guise of Miss Descartes, made a huge graph of petersens, with a nonagon in each placental zone. Neither (s)he nor Isaacs saw that the leads there could be connected as they pleased.

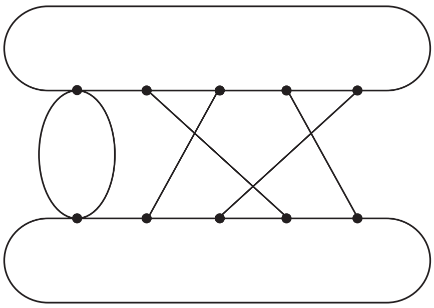
** Isaacs was instinctually correct to use the term 'product'. Referring to Appendix 4 of (2), An algebra for the natural numbers, we see that multiplication is a form of nuclear insertion, since the crossing of the graphic representative of a number can be seen equally as a stripping of its outer shell. The common multiplication of numbers can be seen as denucleating the multiplier and inserting its nucleus into all the links of the multiplicand. In the basic insertion described here, we insert the nucleus into just one of the links of the multiplicand, so the result of the operation can be seen as a partial product of the two graphs. (cont. next page)



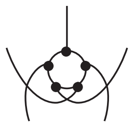
K_1



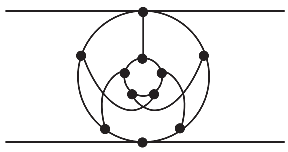
K_2



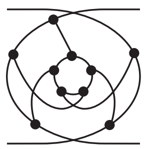
expand



denucleate



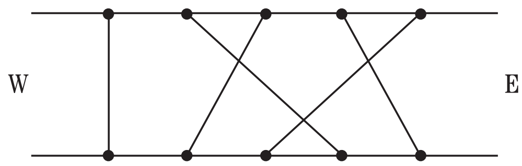
insert



simplify

The efficacy of nuclear insertion can be seen even more clearly by considering a form of ladder graph. We learned in Part II that a straight ladder connexion cannot make a paradox. But if we scramble the rungs it can.

cont. footnote: Thus all graphs can be employed to represent numbers, by counting the number of lines in the graph, and ignoring how they are connected. Then all n-graphs represent multiples of n, and each constant value of n possible to a given number indicates a divisor of that number. A number is prime if and only if its graph is unique, i.e. if the only value possible to n is the number itself. The prime number 2 is also unique in another way, in that its ring component must belong to a degenerate 2-graph and has no nucleus.

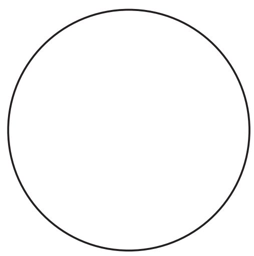


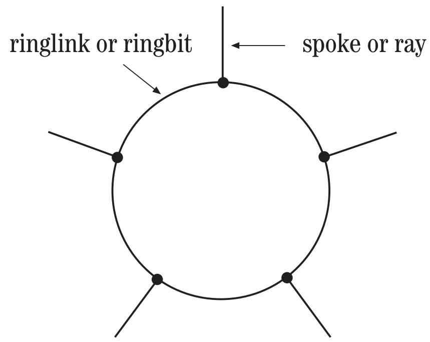
If we connect E to W, it is the petersen P. The scrambled 5-rung ladder is not a proper chain unit, since putting any number in series makes it colorable. But if we take the crisscrossed section, which, suitably colored, I call a trail t (see (2), p 171), we can reduplicate t as often as we like and the result must be uncolorable, since each reduplication is a dot production or nuclear insertion* of another P.

We can see that this case of t, which I call the petersen trail o_1, is basically the only way to do it, and that various connected multiples of 4 scrambled rungs, producing apparently longer connected trails, are all factorable compounds of o_1. See figure 14.2 in Appendix 5 of (2). The other factors in the J's and the unicorn are the only ways we can introduce other elements into a trail without destroying the paradox-making power of o_1.

We now consider the unicorn Q. Isaacs thought there might be other members of this class but couldn't find any. Discarding the already-known members, J_5 and P, it seems astonishing that Q, made by code-eliding two J_5's, should be unique, but I shall prove it is.

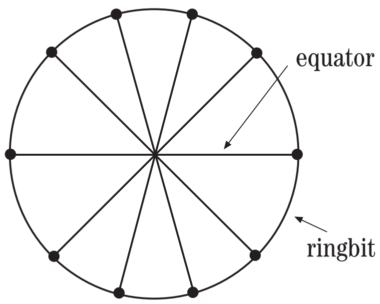
We shall need some definitions.


ring component RC



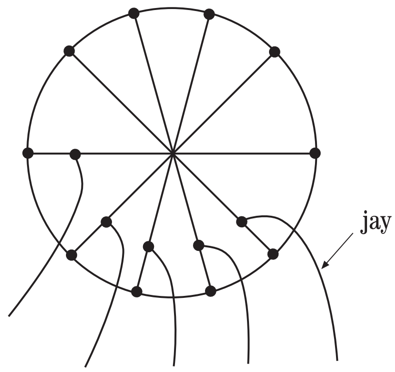
RC + rays = ring

Trail reduplication accounts for a particular class of cases of nuclear insertion, as the reader can readily ascertain by drawing examples. For example take the last picture ('simplify') in my illustration of nuclear insertion and avoid the top and bottom chiasmata x by connecting them >< and we have a trail reduplication.



antiring component AC = sectron

An *equator* has an equal number of ringbits each side.



sectron + jays = antiring or jing

If the colors of the rays are such that the ring can be colored we say they are ring-compatible or *rink*. I f they are such that the ring can not be colored we say they are ring-incompatible or *jink*. We summarize this axiomatically.

Axiom 1

Iff the rays (r_i) *are jink, the ring* R *can not be colored.*

Converse. *Iff* R *can be colored, the* (r_i) *are rink.*

Axiom 2

Iff the jays (j_i) *are rink, the jing* J *can not be colored.*

Converse. *Iff* J *can be colored, the* (j_i) *are jink.*

Call a link-scrambling that stops the coloring of a particular configuration, stop-connected in respect of that configuration.

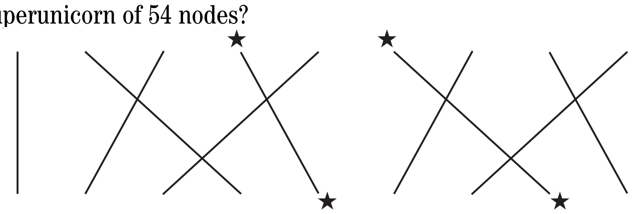
Why the unicorn can exist at all is because of the "accident" that in a 5-rung ladder there can be just two essentially-different color codings, one rink and one jink, so what is not an example of one must be an example of the other.

Thus two 5-jings stop-connected together must, by Axiom 2, be uncolorable, and this is the unicorn Q.

By repeatedly adding connexions in sets of 4, petersen-coded, all rings of size $4n + 1$ can be stop-connected. The question remains, Can we stop-connect two similarly-connected jings together, with $n > 1$, to produce larger uncolorables of the Q class? It looks plausible.

In fact we can't, and the reason is simple: there is basically just one rink-connexion for an uneven-sized ring, with the two odd colors together; but with $n > 1$ there are increasingly more jink-connexions, with the two odd colors apart.

For a simple illustration, we can add a second petersen coding to make a ring-stop coding of 9 links. The question becomes, Can this also act as a jing-stop coding, thus making a superunicorn of 54 nodes?



Putting stars at the ends of links that we can color oddly, we see that stars that are together on one side of the ladder must always come apart on the other, but stars that are apart on one side need not come together on the other. Therefore, although one ring on either side of the ladder-rungs might* not be colorable, both jings must be colorable. The coloring algorithm in all cases is easily extrapolated from my coloring of figure 2.3.

The reader can readily extend this to all ring-stop codings greater than five (there are not many kinds) to see that none of them can be jing-stop. We have thus proved the

Theorem

The unicorn is unique.

So Isaacs found all classes of uncolorable 3-graph, not only in principle, but also in practice. It seems incredible to me that he did not know this, at least subconsciously, however much he protested otherwise.

But consider the climate into which he came. He was not a graph theorist. He came new to a field where uncolorables were not understood. And not just not understood: nobody had even a vestige of comprehension of what they were or how they worked. What he described, without noticing it, was a set of complex higher-degree equations. What I had already proved, and merely had to prove again with respect to this new context, is that no such equation can imbed in the plane.

So what of the four-color conjecture? Everybody knew it was true, so it must always

* This mathematics is full of traps. A ring is uncolorable in this case, but not all codings of more than five rungs that part the odd colors will make an uncolorable ring graph. I leave the reader to work out why. But evidently if either a rink or a jink ordering of the colors can appear on both sides of the ladder, the resulting ring or jing graph can be colored.

have been subliminally obvious. And this is what made it so difficult to prove. To imagine it untrue, we had to resort to the most ridiculous mental contortions. Having found the smallest uncolorable, we then had to ask ourselves, Is there a larger smallest uncolorable? The idiotic nature of the question forces us to consider that we must have gone horribly wrong somewhere.

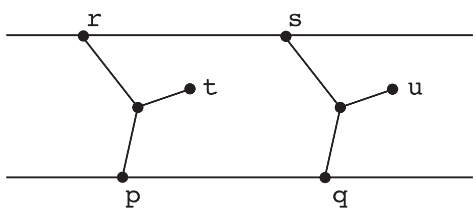
Subliminal awareness goes directly to the heart of the problem, but consciousness invariably takes a long way round before trying anything simple. Thus we all began the hardest possible way, to look for a coloring algorithm and prove it will always work. Next hardest is to take the indirect approach in the plane, and show that uncolorability there is contradictory, which is what I do in (2). It is still long and difficult, but instructive. Easiest of all is to leave the plane altogether and go into the nonplane, where we find, surprise surprise, here are all the uncolorables! It may be considered the neatest of my proofs so far, but its advent is just a little sad, because now we shall all forget what the problem used to look like.

The great tower of the four color problem was a familiar object on the mathematical skyline of all our boyhoods, and not a few girlhoods too. To see it disappear is a bit of a shock, and we may hanker nostalgically after it as we do for the now-demolished landmarks of our childhood haunts. So it is still nice to tread once more the ancient paths of Kempe and Heawood, as I do in Appendix 5 of (2), and take one last look at the skyline before the developers rebuild it.

[*London 1997 07 27 0521*]

A brief summary of my proof of the map theorem in this paper

Consider chains of trivalencies. A least chain must have two units.



Once we have linked the points p, q and r, s, the points t, u are either cut off or cannot be linked to a third distinct such unit in the plane without crossing a line. This proves that all continuous such chains with a unit-count and a linkage between units of more than 2 cannot be planar.

Color may be considered as a token of value, and we cannot make a chain of values paradoxical unless there is a forced change of value from one link to the next in each strand of the linkage between units, and the chain reconnects with itself after an odd number of units. Therefore, the only way chain units could be linked paradoxically (= "uncolorably") is by some such linking. We do not need to know whether any linking that is paradoxical exists. It suffices to know that if there could be a paradoxical linking, it would have to be done this way.

But we have established that doing it this way must be nonplanar.

Therefore, all 3-graphs that can define maps and that are not nonplanar must be non-paradoxical, i.e. 3-colorable. Thus by Tait's and Heawood's theorems all plane maps must be 4-colorable.

Q E D

References

(1) Isaacs, Rufus, Infinite families of nontrivial trivalent graphs which are not tait colorable, *American Math Monthly* 73 (1975) 221-239.

(2) Spencer-Brown, George, *Laws of Form*, International edition, Lübeck and London 1997 (English-language Distributor: Spencer-Brown Publishing, 18A Greville Place London NW6 5JH, telephone/fax +44 (0) 171 624 2358).

18A Greville Place
London NW6 5JH

telephone/fax +44 (0)1985 844 855
other telephones +44 (0)836 313122
+44 (0)1485 544155